

LAW OF LARGE NUMBERS FOR INFINITE RANDOM MATRICES OVER A FINITE FIELD

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To Grigori Olshanski on the occasion of his 65th birthday

ABSTRACT. Asymptotic representation theory of general linear groups $GL(n, F_q)$ over a finite field leads to studying probability measures ρ on the group \mathbb{U} of all infinite uni-uppertriangular matrices over F_q , with the condition that ρ is invariant under conjugations by arbitrary infinite matrices. Such probability measures form an infinite-dimensional simplex, and the description of its extreme points (in other words, ergodic measures ρ) was conjectured by Kerov in connection with nonnegative specializations of Hall–Littlewood symmetric functions.

Vershik and Kerov also conjectured the following Law of Large Numbers. Consider an infinite random matrix drawn from an ergodic measure coming from the Kerov’s conjectural classification and its $n \times n$ submatrix formed by the first rows and columns. The sizes of Jordan blocks of the submatrix can be interpreted as a (random) partition of n , or, equivalently, as a (random) Young diagram $\lambda(n)$ with n boxes. Then, as $n \rightarrow \infty$, the rows and columns of $\lambda(n)$ have almost sure limiting frequencies corresponding to parameters of this ergodic measure.

Our main result is the proof of this Law of Large Numbers. We achieve it by analyzing a new randomized Robinson–Schensted–Knuth (RSK) insertion algorithm which samples random Young diagrams $\lambda(n)$ coming from ergodic measures. The probability weights of these Young diagrams are expressed in terms of Hall–Littlewood symmetric functions. Our insertion algorithm is a modified and extended version of a recent construction by Borodin and the second author [16]. On the other hand, our randomized RSK insertion generalizes a version of the RSK insertion introduced by Vershik and Kerov [71] in connection with asymptotic representation theory of symmetric groups (which is governed by nonnegative specializations of Schur symmetric functions).

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1. INTRODUCTION

In §§1.1–1.3 we describe the setup and our main results, and then in §1.4 and §1.5 we discuss connections with asymptotic representation theory.

1.1. Infinite random matrices over a finite field. Let $q = p^d$ be a prime power, F_q be the corresponding finite field, and $GL(n, F_q)$ be the group of all invertible $n \times n$ matrices over F_q .

Let \mathbb{U} be the group of all infinite uni-uppertriangular matrices over F_q , i.e., matrices $X = [X_{ij}]_{i,j=1}^\infty$ for which $X_{ii} = 1$ and $X_{ij} = 0$ for $i > j$. This is a compact group (under the topology of pointwise convergence of matrix elements).

For any uni-uppertriangular $n \times n$ matrix $g \in GL(n, F_q)$, denote by $Cyl_g \subset \mathbb{U}$ the cylindrical subset consisting of all infinite matrices in \mathbb{U} whose top $n \times n$ corner coincides with g . Note that all eigenvalues of such a matrix g are all equal to 1, and so its conjugacy class in $GL(n, F_q)$ is completely determined by sizes of its Jordan blocks. We will identify sizes of these blocks with partitions λ of n (= Young diagrams λ with $|\lambda| = n$ boxes; see §2.1 for notation).

Definition 1.1. A probability Borel measure ρ on \mathbb{U} is called *central* if for any finite uni-uppertriangular matrix g the measure $\rho(Cyl_g)$ depends only on the conjugacy class of g , i.e., on the partition corresponding to sizes of its Jordan blocks.

Centrality property means conjugation-invariance in the sense that if $h \in GL(\infty, F_q) = \bigcup_{n=1}^\infty GL(n, F_q)$ (i.e., h is an infinite matrix which differs from the identity matrix in a finite number of matrix elements) and $M \subset \mathbb{U}$ is a Borel subset such that $hMh^{-1} \subset \mathbb{U}$, then it must be that $\rho(M) = \rho(hMh^{-1})$.

Central probability measures on \mathbb{U} form a convex set. Its extreme points (i.e., central measures which cannot be expressed as nontrivial convex combinations of other central measures) will be referred to as *ergodic central measures*.

The classification of ergodic central measures on \mathbb{U} is a well-known open problem related to the asymptotic representation theory of the linear groups $GL(n, F_q)$, see §1.5 below for more discussion and references. A conjectural answer to the problem is given by Kerov [44], [45, Ch. 2.9]:¹

Conjecture 1.2 (Kerov). *Ergodic central measures on \mathbb{U} are in one-to-one correspondence with triplets $(\alpha; \beta; \gamma) \in \mathbb{R}^{2\infty+1}$ such that*

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \dots \geq 0), \quad \beta = (\beta_1 \geq \beta_2 \geq \dots \geq 0), \quad \gamma \geq 0, \quad (1.1)$$

and

$$\sum_{i=1}^\infty \alpha_i + \sum_{i=1}^\infty \frac{\beta_i}{1 - q^{-1}} + \frac{\gamma}{1 - q^{-1}} = 1. \quad (1.2)$$

The correspondence is established via the measures of cylindrical sets:

$$\rho^{\alpha; \beta; \gamma}(Cyl_g) = \frac{q^{-n(n-1)/2 + \sum_i (i-1)\lambda_i}}{(1 - q^{-1})^n} Q_\lambda(\alpha; \beta; \mathbf{Pl}_\gamma \mid 0, q^{-1}) \quad (1.3)$$

¹The conjecture was originally formulated in equivalent terms of nonnegative specializations of Hall–Littlewood symmetric functions. Another equivalent formulation involves coherent probability measures on the Young branching graph with formal edge multiplicities depending on q (cf. §5.4). See [6, Thm. 2.3], [34], [37, Prop. 4.7] for details of these equivalences.

for any $n \geq 1$ and any uni-uppertriangular $n \times n$ matrix g , where λ , $|\lambda| = n$, corresponds to Jordan block sizes of g . Here $Q_\lambda(\alpha; \beta; \mathbf{P}_\gamma | 0, \mathfrak{q}^{-1})$ denotes the specialization of the “ Q ” Hall–Littlewood symmetric function, see §2.²

Probability measures $\rho^{\alpha; \beta; \gamma}$ on \mathbb{U} with cylindrical probabilities (1.3) exist and are indeed ergodic, see Gorin–Kerov–Vershik [37, Prop. 4.7].

One example of a central measure is the uniform measure on \mathbb{U} studied by Borodin [5], [6]. This measure is ergodic, it corresponds to taking $\alpha_i = (1 - \mathfrak{q}^{-1})\mathfrak{q}^{1-i}$, $i = 1, 2, \dots$, and setting all β_i ’s and γ to zero (see also §1.5 below for connection to unipotent traces).

Restating (1.3) in terms of the distribution of the Young diagram λ corresponding to Jordan block sizes, one arrives at the probability distribution

$$\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{P}_\gamma}(\lambda) := n! Q_\lambda(\alpha; \beta; \mathbf{P}_\gamma | 0, \mathfrak{q}^{-1}) P_\lambda(\mathbf{P}_1 | 0, \mathfrak{q}^{-1}) \quad (1.4)$$

on the set \mathbb{Y}_n of all Young diagrams with n boxes. Here P_λ is the “ P ” Hall–Littlewood symmetric function,³ and $P_\lambda(\mathbf{P}_1 | 0, \mathfrak{q}^{-1})$ denotes the so-called Plancherel specialization of P_λ (see §2). This Plancherel specialization incorporates the number of uni-uppertriangular matrices from $GL(n, F_\mathfrak{q})$ having the given Jordan blocks sizes determined by λ . The passage from (1.3) to (1.4) follows from Fulman [33, Thm. 1] and Gorin–Kerov–Vershik [37, §4], see also §5.4 below for a connection to the Plancherel specialization and to the Young graph with certain formal edge multiplicities.

Remark 1.3. Because the measures $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{P}_\gamma}$ for various n come from the same distribution $\rho^{\alpha; \beta; \gamma}$ on infinite matrices over $F_\mathfrak{q}$, they satisfy certain *coherency relations* (see §5.4). Note that for various $n = 1, 2, \dots$ the corresponding random Young diagrams $\lambda(n) \in \mathbb{Y}_n$ distributed according to $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{P}_\gamma}$ are defined on *the same probability space* (on which the infinite random matrix is defined).

We refer to Gorin–Kerov–Vershik [37] and Fulman [32], [34], [35] for further connections between random matrices over a finite field and Hall–Littlewood symmetric functions. See also §1.5 below for a brief discussion of asymptotic representation theory of the groups $GL(n, F_\mathfrak{q})$.

1.2. Law of Large Numbers. Our main result is the proof of the following Law of Large Numbers for sizes of Jordan blocks of random infinite uni-uppertriangular matrices over $F_\mathfrak{q}$ under an ergodic central measure:

Theorem 1.4 (Vershik–Kerov’s conjecture [74], [37]). *Let $(\alpha; \beta; \gamma) \in \mathbb{R}^{2\infty+1}$ be any triplet satisfying (1.1)–(1.2) such that the third parameter γ is zero. Let for each $n = 1, 2, \dots$, $\lambda(n) \in \mathbb{Y}_n$ be the random Young diagram corresponding to sizes of Jordan blocks of the $n \times n$ truncation of the random matrix distributed according to $\rho^{\alpha; \beta; 0}$. Then, $\rho^{\alpha; \beta; 0}$ -almost surely,*

$$\frac{\lambda_i(n)}{n} \rightarrow \alpha_i, \quad \frac{\lambda'_i(n)}{n} \rightarrow \frac{\beta_i}{1 - \mathfrak{q}^{-1}}, \quad i = 1, 2, \dots, \quad (1.5)$$

where $\lambda_i(n)$ and $\lambda'_i(n)$ denotes the length of the i -th row (resp. column) of $\lambda(n)$ (see also §2.1).

²Our notation of parameters $(\alpha; \beta; \gamma)$ borrowed from Borodin–Corwin [8] and also used in Borodin–Petrov [16] differs from the one of Kerov [45], see also Gorin–Kerov–Vershik [37]. Details are explained in Remark 2.6 below.

³ P_λ is a constant multiple of Q_λ . We use the standard notation of [50].

In fact, we prove Theorem 1.4 for any value of the parameter $t = \mathfrak{q}^{-1} \in [0, 1]$ in the measures (1.4) (this t is usually referred to as the Hall–Littlewood parameter), not just for an inverse of a prime power. See Theorem 7.1.

Remark 1.5. The Law of Large Numbers implies existence of the asymptotic speeds of particles in the q -PushTASEP (q -deformed pushing totally asymmetric simple exclusion process) with varying particle speeds. This $(1 + 1)$ -dimensional continuous-time integrable particle system was introduced and studied in Borodin–Petrov [16] and Corwin–Petrov [20]. See Remark 7.2 for more detail.

Remark 1.6. Probability measures (1.4) may be viewed as extreme coherent measures on the Young graph with certain formal edge multiplicities related to Hall–Littlewood polynomials (cf. Remarks 1.3 and 1.15). In view of this connection, one would expect these measures to have asymptotic frequencies (as it happens for extreme coherent measures on other branching graphs, e.g., see [68], [43], [70], [56], [36], [14], [60]). Our Theorem 1.4 is exactly a statement about these asymptotic frequencies. See also Conjecture 1.9 below for the case of Macdonald edge multiplicities.

Let us now formulate three conjectures related to Theorem 1.4.

Conjecture 1.7 (Case $\gamma > 0$). *We believe that the technical assumption $\gamma = 0$ can be dropped, and the same convergence (1.5) could be established for any triplet $(\alpha; \beta; \gamma)$ with (1.1)–(1.2).*

Conjecture 1.8 (Central Limit Theorem). *If the α - and the β -parameters are distinct (when they are positive), i.e., $\alpha_1 > \alpha_2 > \dots$ and $\beta_1 > \beta_2 > \dots$, then the lengths of rows and columns of random Young diagrams $\lambda(n)$ satisfy a Central Limit Theorem: The infinite vector $\{\lambda_1(n), \lambda_2(n), \dots; \lambda'_1(n), \lambda'_2(n), \dots\}$ is asymptotically jointly Gaussian after subtracting the limiting means (1.5) and normalizing by \sqrt{n} . The limiting covariances are equal to:⁴*

- $\alpha_i \mathbf{1}_{i=j} - \alpha_i \alpha_j$ between λ_i and λ_j ;
- $\frac{\beta_i}{1 - \mathfrak{q}^{-1}} \mathbf{1}_{i=j} - \frac{\beta_i \beta_j}{(1 - \mathfrak{q}^{-1})^2}$ between λ'_i and λ'_j ;
- $-\frac{\alpha_i \beta_j}{1 - \mathfrak{q}^{-1}}$ between λ_i and λ'_j .

One can also replace \mathfrak{q}^{-1} by any value of the Hall–Littlewood parameter $t \in [0, 1]$ in the formulation of Conjecture 1.8. In the case of symmetric groups (corresponding to $t = 0$, see §1.4 below) a similar Central Limit Theorem was established by Feray and Meliot [25], [51] and Bufetov [19]. In this case, the behavior of fluctuations changes when the assumptions on parameters $(\alpha; \beta; \gamma)$ are not satisfied. This suggests the same restrictions on parameters for the $t > 0$ case as well.

Conjecture 1.8 should be accessible by the technique of the present paper: The main idea behind our proof of Theorem 1.4 is that the asymptotic behavior of random Young diagrams is shown to be the same as the asymptotic behavior of random words of fixed length with independently distributed letters (see §1.3 below). Conjecture 1.8 asserts that the asymptotic behavior of fluctuations coincides as well, so the covariance matrix for these two models should be the same. However, we do not pursue this direction here.

⁴Here and below $\mathbf{1}_A$ means the indicator of A .

For the uniform measure on \mathbb{U} , the Law of Large Numbers (Theorem 1.4) and the Central Limit Theorem (Conjecture 1.8) were established by Borodin [5], [6].

One can replace the Hall–Littlewood symmetric functions by the Macdonald ones which depend on two parameters $q, t \in [0, 1)$, see §2.2 (note the difference between the Macdonald parameter q and the prime power $\mathfrak{q} = t^{-1}$ which is the size of the base finite field). The corresponding probability measures $\mathcal{M}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}$ on Young diagrams with n boxes can be defined similarly to (1.4). These measures with Macdonald parameters were introduced by Fulman [31] and studied in great detail by Borodin and Corwin [8]. See also Forrester–Rains [30].

Conjecture 1.9 (Law of Large Numbers with Macdonald parameters). *Let $\lambda(n)$ be the random Young diagram distributed according to $\mathcal{M}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}$. Then, with almost sure convergence,*

$$\frac{\lambda_i(n)}{n} \rightarrow \alpha_i, \quad \frac{\lambda'_i(n)}{n} \rightarrow \beta_i \frac{1-q}{1-t}, \quad i = 1, 2, \dots \quad (1.6)$$

1.3. Randomized Robinson–Schensted–Knuth (RSK) insertion. Our main technique for studying probability measures $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}$ (1.4) and proving Theorem 1.4 is a certain new sampling algorithm for these measures. Namely, we introduce a randomized version of the classical Robinson–Schensted–Knuth (RSK) insertion algorithm which samples $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}$. About the classical RSK, e.g., see Stanley [65, Ch. 7], Sagan [62], and also Borodin–Petrov [16, §7].

Let us describe this insertion algorithm in the case when there are only finitely many nonzero α parameters, namely, $\alpha = (\alpha_1, \dots, \alpha_N)$, and that all β_i and γ are zero. Then (1.2) means that $\alpha_1 + \dots + \alpha_N = 1$. The case of general parameters is described in §6. The input of the algorithm is a word $w = \xi_1 \xi_2 \dots \xi_n$, where $\xi_i \in \mathcal{A} = \{1, 2, \dots, N\}$. Applied to a fixed word w , the algorithm produces a *random* interlacing integer array (see Fig. 1)

$$\{\lambda_i^{(m)} : m = 1, \dots, N, i = 1, \dots, m\}, \quad \lambda_{i+1}^{(m)} \leq \lambda_i^{(m-1)} \leq \lambda_i^{(m)}.$$

In an interlacing array we call a particle $\lambda_i^{(m)}$ *blocked* if $\lambda_i^{(m)} = \lambda_{i-1}^{(m-1)}$. Otherwise the particle is called *free*.

The interlacing array can be interpreted as a semistandard Young tableau \mathcal{P} of shape $\lambda^{(N)} = (\lambda_1^{(N)} \geq \dots \geq \lambda_N^{(N)})$. For general parameters, the alphabet \mathcal{A} is different, and the notion of a semistandard Young tableau \mathcal{P} has to be changed accordingly. Such more general \mathcal{A} -tableaux first appeared in the work of Vershik and Kerov [71], see also Berele and Regev [4]. (For definitions of standard tableaux, semistandard tableaux, and \mathcal{A} -tableaux see Definition 6.11 and Remark 6.12.)

The randomized insertion algorithm starts from the empty configuration, i.e., $\lambda_i^{(m)} = 0$ for all m and i . Letters from the word $w = \xi_1 \dots \xi_n$ arrive one by one, and are inserted into the semistandard tableau \mathcal{P} . When a letter ξ_j is inserted, the interlacing array (corresponding to \mathcal{P} as on Fig. 1) undergoes the following modifications:

- First, at the level ξ_j the leftmost free particle moves to the right by one.
- After that, modifications propagate upwards to all levels $m = \xi_j, \xi_j + 1, \dots, N$ as follows:
 - If a particle $\lambda_i^{(m)}$ moves to the right by one, and $\lambda_i^{(m)} = \lambda_i^{(m+1)}$ before the move, then the particle $\lambda_i^{(m+1)}$ also immediately moves to the right by one with probability one. This second move restores the interlacing which was broken by the first move (*mandatory short-range pushing*).

1	1	1	1	1	2	3	3	3	4
2	2	4	4	4	4				
3	3								
4									

1 2 6 10
 2 2 9
 2 6
 5

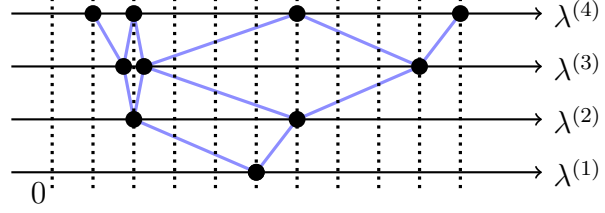


FIGURE 1. A semistandard Young tableau \mathcal{P} and the corresponding interlacing integer array of depth $N = 4$. The shape of the Young tableau is $\lambda^{(N)} = (10, 6, 2, 1)$. Particles (on the right) are located at positions $\lambda_i^{(m)}$, where m and i represent vertical and horizontal coordinates, respectively. When there are several particles occupying the same position, we draw them close to each other. Zigzags indicate the interlacing property.

- Otherwise, if a particle $\lambda_i^{(m)}$ has moved (to the right by one), then at the next level $m + 1$ the first free upper right neighbor of $\lambda_i^{(m)}$ immediately moves to the right by one with probability $r_i(\lambda^{(m)}, \lambda^{(m+1)} | \mathbf{q}^{-1})$ (*pushing*), or the upper left neighbor $\lambda_{i+1}^{(m+1)}$ moves with the complementary probability $1 - r_i(\lambda^{(m)}, \lambda^{(m+1)} | \mathbf{q}^{-1})$ (*pulling*). These probabilities depend on \mathbf{q} and on the number of particles at levels m and $m + 1$ which occupy the horizontal position of $\lambda_i^{(m)}$ before its move. They are determined as on Fig. 2 (see §6.2 for a complete description).

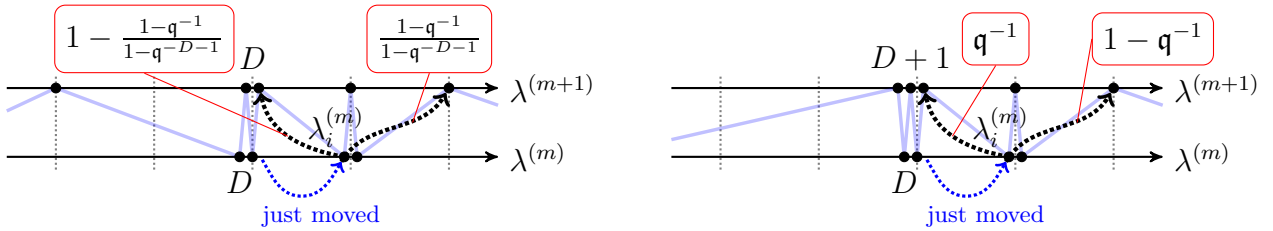


FIGURE 2. Pushing and pulling probabilities in the sampling algorithm (note that the number D can be zero).

For example, if the current state of the interlacing array is as on Fig. 1 and the next inserted letter is $\xi_{20} = 2$, then the result of this insertion will be distributed as follows (values that

changed are framed):

$$\begin{array}{|c|} \hline \begin{array}{c} 1 \quad 2 \quad \boxed{7} \quad 10 \\ 2 \quad \boxed{3} \quad 9 \\ \boxed{3} \quad 6 \\ 5 \end{array} \\ \hline \end{array} \quad \text{with prob. } \frac{1}{1 + \mathfrak{q}^{-1}}, \quad \begin{array}{|c|} \hline \begin{array}{c} 1 \quad \boxed{3} \quad 6 \quad 10 \\ 2 \quad \boxed{3} \quad 9 \\ \boxed{3} \quad 6 \\ 5 \end{array} \\ \hline \end{array} \quad \text{with prob. } \frac{\mathfrak{q}^{-1}}{1 + \mathfrak{q}^{-1}}.$$

Indeed, the propagation of the move from level 2 to level 3 is a mandatory pushing, and from level 3 to level 4 the pushing or pulling probability is determined as on Fig. 2 (left), with $D = 1$. See also §6.6 for another example involving general parameters.

Theorem 1.10. *If the randomized RSK insertion algorithm is applied to a random word $w = \xi_1 \xi_2 \dots \xi_n$ with independent letters $\xi_i \in \{1, 2, \dots, N\}$ such that $P(\xi_i = k) = \alpha_k$, then the distribution of the top row $\lambda^{(N)}$ of the array is exactly $\mathcal{HL}_n^{\alpha; \mathbf{0}; \mathbf{P} \mathbf{1}_0}$, i.e., the measure (1.4) with the parameters $\alpha = (\alpha_1, \dots, \alpha_N)$.*

We prove this theorem along with the corresponding statement for general parameters in §6.5 (see Theorem 6.14). It implies that the measures (1.4) we are interested in are images of Bernoulli measures on words under a certain randomized algorithm (this property is the same for general parameters). Analyzing this algorithm, we prove that the main contributions to lengths of rows and columns under (1.4) come from the Bernoulli measure part, and thus arrive at the Law of Large Numbers (Theorem 1.4).

Let us now make a number of remarks on our randomized RSK insertion algorithm.

Remark 1.11 (Discrete analogue of Dyson's Brownian motion). In the course of our randomized RSK insertion algorithm, the top row $\lambda^{(N)}$ (i.e., the shape of the corresponding Young tableau) evolves according to a certain Markov chain which first appeared in Fulman [31]. This process (which we describe in §4.2) can be viewed as a certain discrete analogue of the Dyson's Brownian motion [22].

Remark 1.12 (Symmetric groups). If the Hall–Littlewood parameter t (which is equal to \mathfrak{q}^{-1} for the purposes of studying random matrices over $F_{\mathfrak{q}}$) is set to zero, then the randomized insertion becomes *deterministic*, and coincides with the classical RSK insertion with column insertion. We discuss the $t = 0$ degeneration in detail in §1.4 below.

Remark 1.13 (Sampling algorithms for measures involving Hall–Littlewood functions). In the previous years, various (probabilistic) algorithms were constructed for sampling probability measures related to Hall–Littlewood symmetric functions (such as our measures $\mathcal{HL}_n^{\alpha; \beta; \mathbf{P} \mathbf{1}_\gamma}$). Analyzing these algorithms, one manages to extract certain specific properties of the sampled probability measures. A sampling algorithm for random Young diagrams corresponding to the uniform measure on \mathbb{U} was introduced by Kirillov [48] and studied by Borodin [5], [6]. It is well-adapted to proving Theorem 1.4 for this uniform measure.

There are also sampling algorithms constructed by Fulman [32], [34], which allow to obtain information about certain other probability measures on Young diagrams related to Hall–Littlewood symmetric functions. Namely, it is possible to derive (in some form) distributions of observables of random Young diagrams such as their row or column lengths. See also Fulman [35] for a recent connection of measures involving Hall–Littlewood symmetric functions to the Cohen–Lenstra heuristics of Number Theory.

Remark 1.14 (Randomized RSK insertions). Other randomized RSK insertion algorithms were also developed and studied by O’Connell and Pei [55], [58]. The latter paper also explains the randomized RSK insertion algorithms in a more traditional language of Young tableaux and Fomin’s growth diagrams (about the latter see Fomin [26], [27], [28], [29]).

A family of algorithms for sampling the measures $\mathcal{H}\mathcal{L}_n^{\alpha;0;\mathbf{Pl}_0}$ was constructed in Borodin–Petrov [16]. A priori these algorithms involved negative transition probabilities. The sampling algorithm described above is present in [16] in a hidden form: it is singled out by requiring nonnegative transition probabilities. In §§4–5 we also generalise the mechanism of [16] to sample measures $\mathcal{H}\mathcal{L}_n^{\alpha;\beta;\mathbf{Pl}_\gamma}$. The passage from parameters α to parameters β is possible via a certain duality related to the transposition of Young diagrams, see §5.3. The Plancherel part \mathbf{Pl}_γ can be added by considering interlacing particle arrays with “continuous floors”, cf. §5.5. Our sampling algorithm for the measures $\mathcal{H}\mathcal{L}_n^{\alpha;\beta;\mathbf{Pl}_\gamma}$ lives on interlacing arrays generalizing the ones on Fig. 1, see Fig. 11 in §6.

Remark 1.15 (Branching graphs). It is worth noting that the algorithm of Borodin which samples measures (1.4) for special parameters (Remark 1.13), differs significantly from our construction. Namely, the former employs the coherency property on the Young graph with Hall–Littlewood formal edge multiplicities which corresponds to adding one box to a Young diagram (cf. Remarks 1.3, 1.6, and §5.4). Moreover, this algorithm can be viewed as a univariate dynamics of §4.2.

On the other hand, our construction benefits from a connection to Gelfand–Tsetlin graph with another type of branching corresponding to adding horizontal strips to a Young diagram (with edge multiplicities also related to Hall–Littlewood symmetric functions). One can say that we work with Markov dynamics on paths in the Gelfand–Tsetlin graph (these are interlacing particle configurations), and our constructions are in the spirit of dynamics on interlacing particle arrays of Borodin–Ferrari [10], Borodin [7], Borodin–Olshanski [13], Borodin–Corwin [8] (they are based on an idea of Diaconis–Fill [21]), and more general multivariate dynamics developed in Borodin–Petrov [16].

The interplay between the coherency property on the Young graph and its connections to the Gelfand–Tsetlin graph was employed in, e.g., Borodin–Gorin [11]. Representation-theoretic consequences of connections between the Young and Gelfand–Tsetlin graphs are discussed in Borodin–Olshanski [15].

1.4. Symmetric groups. Here and in the next subsection we briefly summarize representation-theoretic constructions which lead to the classification problem of §1.1. We start with an analogous problem for symmetric groups. This part of the introduction is not essential for understanding our main results and constructions.

One of the central problems of the asymptotic representation theory of symmetric groups $S(n)$ is to classify *irreducible characters* of the infinite symmetric group $S(\infty) = \bigcup_{n=1}^{\infty} S(n)$. Elements of $S(\infty)$ are permutations of the infinite set $\{1, 2, \dots\}$ which move only finitely many numbers. A character of $S(\infty)$ is a positive definite central function χ on $S(\infty)$ which is normalized by $\chi(e) = 1$. Characters of $S(\infty)$ form a convex set, and irreducible characters are (by definition) extreme points of this set. Irreducible characters correspond to finite factor representations of $S(\infty)$, e.g., see [68].

Theorem 1.16 (Edrei [23]⁵ and Thoma [66]). *Irreducible characters of $S(\infty)$ are in one-to-one correspondence with triplets $\tilde{\mathbf{A}} := (\tilde{\alpha}; \tilde{\beta}; \tilde{\gamma}) \in \mathbb{R}^{2\infty+1}$ such that*

$$\tilde{\alpha} = (\tilde{\alpha}_1 \geq \tilde{\alpha}_2 \geq \dots \geq 0), \quad \tilde{\beta} = (\tilde{\beta}_1 \geq \tilde{\beta}_2 \geq \dots \geq 0), \quad \tilde{\gamma} \geq 0, \quad (1.7)$$

and

$$\sum_{i=1}^{\infty} \tilde{\alpha}_i + \sum_{i=1}^{\infty} \tilde{\beta}_i + \tilde{\gamma} = 1. \quad (1.8)$$

The correspondence is established by restricting χ to the subgroup $S(n) \subset S(\infty)$ permuting the first n numbers. The restriction $\chi|_{S(n)}$ can be decomposed into a convex combination of normalized irreducible characters of $S(n)$ which are indexed by Young diagrams $\lambda \in \mathbb{Y}_n$:

$$\chi|_{S(n)} = \sum_{\lambda \in \mathbb{Y}_n} \mathcal{S}_n^{\tilde{\mathbf{A}}}(\lambda) \frac{\chi_\lambda}{\dim \chi_\lambda},$$

and the coefficients of this combination (this is a probability measure on \mathbb{Y}_n) are

$$\mathcal{S}_n^{\tilde{\mathbf{A}}}(\lambda) = n! s_\lambda(\tilde{\mathbf{A}}) s_\lambda(\mathbf{P}\mathbf{1}_1), \quad (1.9)$$

where s_λ is the Schur symmetric function, and the measure above is given as the product of two specializations of s_λ (see definitions in §2).

The problem of classifying irreducible characters of $S(\infty)$ can be also formulated in equivalent terms of nonnegative specializations of Schur symmetric functions [69], [68], see also [46].

Measures (1.9) also satisfy a certain coherency property on the Young graph, see §5.4. At the level of formulas (1.9) and (1.4), Theorem 1.16 is the degeneration of Conjecture 1.2 when the parameter $t = \mathfrak{q}^{-1}$ is set to zero (then the Hall–Littlewood P_λ and Q_λ both become the Schur function s_λ).

Remark 1.17. For $t = 0$, the randomized RSK insertion algorithm we develop is deterministic, it was introduced by Vershik and Kerov [71] in connection with Theorem 1.16 (and further exploited in, e.g., Sniady [64], see also Romik–Sniady [61]). The Law of Large Numbers in this setting (i.e., an analogue of Theorem 1.4 for $t = \mathfrak{q}^{-1}$ being zero) was obtained earlier also by Vershik and Kerov [69], by a direct investigation of measures (1.9) (they are simpler than (1.4) in that they admit more direct explicit formulas). Bufetov [19] used the deterministic RSK insertion algorithm of [71] to establish a corresponding Central Limit Theorem (Conjecture 1.8 for $t = 0$).

Remark 1.18. We also note that a problem of classifying ergodic conjugation-invariant measures on Hermitian matrices over the *complex numbers* (instead of $F_{\mathfrak{q}}$ as in Conjecture 1.2) was considered by Olshanski and Vershik [57]. This setup is also deeply related to Schur symmetric functions. Moreover, it arises as a degeneration in a certain sense of the problem coming from asymptotic representation theory of unitary groups (over complex numbers). About the latter problem, see Edrei [24] (and also Aissen–Edrei–Schoenberg–Whitney [1], Aissen–Schoenberg–Whitney [2]), Voiculescu [75], Vershik–Kerov [70], Boyer [18], Okounkov–Olshanski [56], Borodin–Olshanski [14], Petrov [60], Gorin–Panova [38].

⁵See also Aissen–Edrei–Schoenberg–Whitney [1].

1.5. Asymptotic representation theory of linear groups over a finite field. The desire to construct a meaningful asymptotic representation theory of the groups $GL(n, F_q)$ (which is in some sense a deformation of the corresponding theory for symmetric groups; the latter one was briefly described in §1.4) leads to considering various groups of infinite matrices over F_q which play the role of a natural $n = \infty$ analogue of the groups $GL(n, F_q)$. A direct analogue of $S(\infty)$, the group $GL(\infty, F_q)$ (see the discussion after Definition 1.1), in fact leads to a poor representation theory, see Thoma [67] and Skudlarek [63].

First example of a “right” $n = \infty$ analogue is the group \mathbb{GLB} (see Vershik and Kerov [73], and also Vershik’s historical preface in [37]) of all invertible almost upper-triangular matrices over F_q . Namely, \mathbb{GLB} consists of all matrices $X = [X_{ij}]_{i,j=1}^\infty$ whose upper $n \times n$ corner is invertible for a large enough n , and, moreover, $X_{ij} = 0$ for $i > j$ and $i > n$, and $X_{ii} \neq 0$ for $i > n$.

A very similar representation theory arises for another group, \mathbb{GLU} , which consists of all matrices $X = [X_{ij}]_{i,j=1}^\infty \in \mathbb{GLB}$ for which $X_{ii} = 1$ for large enough i . For both groups, there is a natural notion of characters which are traces of the so-called Schwartz-Bruhat algebra of the group. Principal (unipotent) extreme traces of this Schwartz-Bruhat algebra are parametrized by the same triplets $(\tilde{\alpha}; \tilde{\beta}; \tilde{\gamma}) \in \mathbb{R}^{2\infty+1}$ satisfying (1.7)–(1.8) as for the infinite symmetric group, see [73], [37, Thm. 2.24]. A posteriori, when the classification is known, these unipotent extreme traces are identified with extreme traces of the infinite-dimensional Iwahori-Hecke algebra $\mathcal{H}_\infty(\mathfrak{q})$. Traces of the latter were classified in Vershik–Kerov [72] and Meliot [51, §7]. See also [37, §3.3] for the identification of two classifications.

Let us now make connection of this classification of extreme unipotent traces to Conjecture 1.2 which is open. By [37, Theorems 4.2 and 4.6], to every unipotent trace of \mathbb{GLU} indexed by $(\tilde{\alpha}; \tilde{\beta}; \tilde{\gamma})$ as above corresponds a unique central probability measure on $\mathbb{U} \subset \mathbb{GLU}$ (Definition 1.1). Moreover, this central probability measure is ergodic, and it is indexed by parameters

$$\{\alpha_r\}_{r=1}^\infty = \{\tilde{\alpha}_i(1 - \mathfrak{q}^{-1})\mathfrak{q}^{1-j}\}_{i,j=1}^\infty, \quad \beta_i = \tilde{\beta}_i(1 - \mathfrak{q}^{-1}), \quad \gamma = \tilde{\gamma}(1 - \mathfrak{q}^{-1})$$

in the sense of Conjecture 1.2. (Note that this transformation is *different* from the more straightforward reparametrization described in Remark 2.6 below.)

We refer to Vershik–Kerov [73] and Gorin–Kerov–Vershik [37] for further details and connections to asymptotic representation theory.

1.6. Outline of the paper. In §2 and §3 we recall necessary objects related to Young diagrams and Macdonald (and Hall–Littlewood) symmetric functions. In particular, in §3 we discuss our main object: *coherent measures* on Young diagrams related to Macdonald symmetric functions (this is a generalization of the measures (1.4)). In §4 we recall and extend the general formalism of [16] for constructing Markov dynamics which map coherent measures onto each other. In §5 and §6 we construct our randomized RSK insertion algorithm for sampling coherent measures. In §7 we employ this sampling algorithm to prove the Law of Large Numbers.

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2. PRELIMINARIES

2.1. Young diagrams. Let \mathbb{Y} denote the set of all *partitions*, i.e., integer sequences of the form $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\lambda)} > 0)$, where $\lambda_i \in \mathbb{Z}_{\geq 0}$. We always identify partitions with *Young diagrams* as in [50, I.1], see also Fig. 3. The number $\ell(\lambda)$ of nonzero components of λ is called the *length* of the partition. Also, let $|\lambda| := \sum_{i=1}^{\ell(\lambda)} \lambda_i$ be the number of boxes in the corresponding Young diagram. When needed, we will append partitions by zeroes, and identify λ with $(\lambda_1, \dots, \lambda_{\ell(\lambda)}, 0, 0, \dots)$. The empty partition is denoted by $\emptyset = (0, 0, \dots)$. For $n \geq 0$, let $\mathbb{Y}_n := \{\lambda \in \mathbb{Y} : |\lambda| = n\}$ be the set of Young diagrams with n boxes.

For two Young diagrams μ, λ such that $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for all $i = 1, \dots, \ell(\mu)$, we will write $\mu \subseteq \lambda$. In this case, the set difference of the diagram λ and the diagram μ is denoted by λ/μ and called a *skew Young diagram*.

If $\mu \subseteq \lambda$ and, moreover,

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \lambda_{\ell(\lambda)-1} \geq \mu_{\ell(\mu)} \geq \lambda_{\ell(\lambda)} \quad (2.1)$$

(this implies that $\ell(\lambda) = \ell(\mu)$ or $\ell(\lambda) = \ell(\mu) + 1$), then we say that the diagram λ is obtained from μ by adding a *horizontal strip* (or, equivalently, that the skew diagram λ/μ is a *horizontal strip*), and denote this by $\mu \prec_h \lambda$.

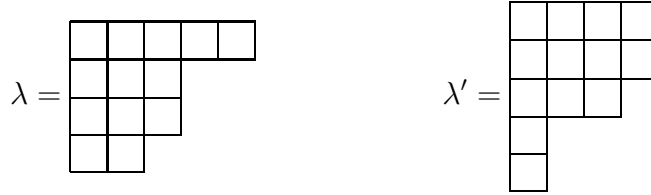


FIGURE 3. Young diagram $\lambda = (5, 3, 3, 2)$ and its transpose $\lambda' = (4, 4, 3, 1, 1)$.

If $\lambda \in \mathbb{Y}$ is represented by a Young diagram, then, reflecting it with respect to the main diagonal, one gets the *transposed diagram* (see Fig. 3).

We say that two diagrams $\mu \subseteq \lambda$ differ by a *vertical strip* (equivalently, that the skew diagram λ/μ is a *vertical strip*) and denote this by $\mu \prec_v \lambda$, iff the transposed diagrams $\mu' \subseteq \lambda'$ differ by a horizontal strip.

For a Young diagram λ , let $\mathcal{U}(\lambda)$ (respectively, $\mathcal{D}(\lambda)$) denote the set of all boxes that can be added to (respectively, removed from) the diagram λ in such a way that the result is *again* a Young diagram (see Fig. 4). For $\lambda, \nu \in \mathbb{Y}$, we will write $\lambda \nearrow \nu$ if ν is obtained from λ by adding a box. Note that adding a box is a very particular case of adding a horizontal (or vertical) strip.⁶ The operation of adding a box will be also denoted as $\nu = \lambda + \square$, or, equivalently, as $\lambda = \nu - \square$.

⁶However, note that a horizontal or a vertical strip is allowed to be empty.

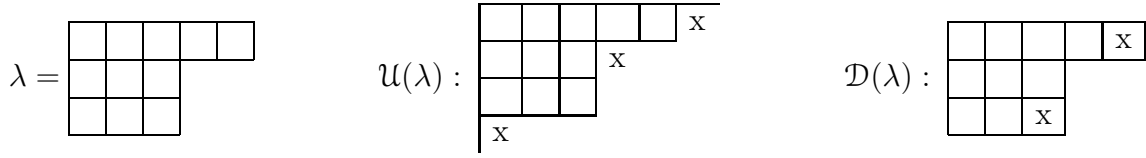


FIGURE 4. Boxes that can be added to (or removed from) a Young diagram.

2.2. Macdonald symmetric functions. Probability measures we consider in the present paper are described in terms of nonnegative specializations of Macdonald (and, in particular, Hall–Littlewood) symmetric functions. Let us briefly recall the necessary definitions. We refer to [50] and [8, §2] for details.

By **Sym** denote the *algebra of symmetric functions* over \mathbb{R} [50, I.2]. It is a commutative algebra $\mathbb{R}[p_1, p_2, \dots]$ generated by 1 and by the (algebraically independent) Newton power sums

$$p_k(x_1, x_2, \dots) = \sum_{i=1}^{\infty} x_i^k, \quad k = 1, 2, \dots$$

Products of power sums $p_\lambda := p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_{\ell(\lambda)}}$, where $\lambda \in \mathbb{Y}$ (with the agreement $p_\emptyset = 1$), form a linear basis in **Sym**. (All linear bases in **Sym** that we consider will be indexed by the set \mathbb{Y} .) The algebra **Sym** possesses a natural grading which is defined by setting $\deg p_k = k$, $k = 1, 2, \dots$

Remark 2.1. Alternatively, each element of **Sym** may be viewed as a symmetric formal power series in x_1, x_2, \dots in which degrees of all monomials are bounded. If in a symmetric function $f(x_1, x_2, \dots)$ all but finitely many (say, N) of the variables are set to zero, then we get a usual symmetric polynomial $f(x_1, \dots, x_N) = f(x_1, \dots, x_N, 0, 0, \dots)$ in finitely many variables.

Moreover, every $f \in \mathbf{Sym}$ can be understood as a sequence of symmetric polynomials f_N in N variables, $N = 1, 2, \dots$, such that $\sup_N \deg f_N < \infty$ and the polynomials $\{f_N\}$ are compatible in the sense that $f_{N+1}(x_1, \dots, x_N, 0) = f_N(x_1, \dots, x_N)$.⁷

A remarkable two-parameter family of linear bases in **Sym** is formed by the *Macdonald symmetric functions* [50, VI]. Let $q, t \in [0, 1)$. Consider a bilinear scalar product $\langle \cdot, \cdot \rangle_{q,t}$ in **Sym** defined on $\{p_\lambda\}$ by

$$\langle p_\lambda, p_\mu \rangle_{q,t} := \mathbf{1}_{\lambda=\mu} z_\lambda(q, t), \quad z_\lambda(q, t) := \left(\prod_{i \geq 1} i^{m_i} (m_i)! \right) \cdot \left(\prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \right),$$

where $\lambda = (1^{m_1} 2^{m_2} \dots)$ means that λ has m_1 parts equal to 1, m_2 parts equal to 2, etc.

Definition 2.2. The *Macdonald symmetric functions* $P_\lambda(\mathbf{x} | q, t)$ (where $\mathbf{x} = (x_1, x_2, \dots)$ and λ runs over all partitions) form a unique family of homogeneous symmetric functions such that:

(1) The functions $\{P_\lambda\}$ are pairwise orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle_{q,t}$.

⁷This means that **Sym** is the *projective limit* (in the category of graded algebras) of algebras of symmetric polynomials in growing number of variables.

(2) For every λ , we have

$$P_\lambda(\mathbf{x} | q, t) = x_1^{\lambda_1} \dots x_{\ell(\lambda)}^{\lambda_{\ell(\lambda)}} + \text{lower monomials in lexicographic order.}$$

The dependence on the parameters (q, t) is in coefficients of the lexicographically lower monomials.⁸

When this does not lead to a confusion, we will omit the notation (q, t) , and simply write $P_\lambda(\mathbf{x})$ or P_λ instead of $P_\lambda(\mathbf{x} | q, t)$.

Also define $Q_\lambda := P_\lambda / \langle P_\lambda, P_\lambda \rangle_{q,t}$, so that the functions P_λ and Q_μ are orthonormal.

In view of Remark 2.1, one can also speak about the Macdonald symmetric polynomials $P_\lambda(x_1, \dots, x_N | q, t)$. They can be alternatively defined as eigenfunctions of certain q -difference operators [50, VI.3].

There are several important special cases of the parameters (q, t) . We are mainly interested in one of them corresponding to setting the first parameter q to zero. Then the Macdonald symmetric functions become the *Hall–Littlewood symmetric functions* [49], [50, III]. If one further sets $t = 0$ (or, equivalently, takes the Macdonald symmetric functions with $q = t$), then one gets the *Schur symmetric functions*. In contrast with the general Macdonald case, both the Hall–Littlewood and Schur symmetric functions admit rather explicit formulas (see I.(3.1) and III.(2.1) in [50], respectively), but we will not use them. See also §2.4 for other interesting particular cases of the Macdonald parameters (q, t) .

Definition 2.3. A *skew Macdonald symmetric function* $Q_{\lambda/\mu}$ indexed by $\mu, \lambda \in \mathbb{Y}$ is defined as the only symmetric function such that $\langle Q_{\lambda/\mu}, P_\nu \rangle_{q,t} = \langle Q_\lambda, P_\mu P_\nu \rangle_{q,t}$ for all $\nu \in \mathbb{Y}$. The P version is then defined through $Q_{\lambda/\mu}$ as $P_{\lambda/\mu} := \frac{\langle P_\lambda, P_\lambda \rangle_{q,t}}{\langle P_\mu, P_\mu \rangle_{q,t}} Q_{\lambda/\mu}$. Skew functions vanish unless $\mu \subseteq \lambda$. One also has $P_{\lambda/\emptyset} = P_\lambda$ and $Q_{\lambda/\emptyset} = Q_\lambda$.

2.3. Specializations of Sym. By a *specialization* of the algebra \mathbf{Sym} we mean an algebra homomorphism $\mathbf{A}: \mathbf{Sym} \rightarrow \mathbb{R}$. Such a map is completely determined by its values $\mathbf{A}(p_k)$ on the power sums. The *trivial* specialization \emptyset is defined as taking value 1 at the constant function $1 \in \mathbf{Sym}$ and sending all the power sums p_k , $k \geq 1$, to zero.

For two specializations \mathbf{A}_1 and \mathbf{A}_2 , we define their *union* $\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2)$ (sometimes we will also use the notation $\mathbf{A}_1 \cup \mathbf{A}_2$) as the specialization defined on power sums as

$$p_k(\mathbf{A}_1, \mathbf{A}_2) = p_k(\mathbf{A}_1) + p_k(\mathbf{A}_2), \quad k \geq 1.$$

If \mathbf{A} is a specialization, define its *multiple* $a \cdot \mathbf{A}$ (where $a \in \mathbb{R}$) by requiring that on homogeneous functions $f \in \mathbf{Sym}$, $f(a \cdot \mathbf{A}) = a^{\deg f} f(\mathbf{A})$.

Important examples of specializations are the so-called *finite length specializations* $\mathbf{A}_{y_1, \dots, y_N}$, where $y_1, \dots, y_N \in \mathbb{R}$, defined as follows. For $f \in \mathbf{Sym}$, let f_N be the corresponding symmetric polynomial in N variables (see Remark 2.1). The image of f under $\mathbf{A}_{y_1, \dots, y_N}$ is

$$f \mapsto f_N(y_1, \dots, y_N). \quad (2.2)$$

The finite length specializations suggest the notation: For $f \in \mathbf{Sym}$ and a specialization \mathbf{A} we will write $f(\mathbf{A})$ instead of $\mathbf{A}(f)$. For finite length specializations we will use a more intuitive notation $f(y_1, \dots, y_N)$ instead of $f(\mathbf{A}_{y_1, \dots, y_N})$.

⁸Lexicographic order means that, for example, x_1^2 is higher than $\text{const} \cdot x_1 x_2$ which is in turn higher than $\text{const} \cdot x_2^2$.

Definition 2.4. A specialization \mathbf{A} of Sym is said to be (q, t) -nonnegative⁹ if $P_{\lambda/\mu}(\mathbf{A} \mid q, t) \geq 0$ for any partitions $\lambda, \mu \in \mathbb{Y}$. The set

$$\mathbb{Y}(\mathbf{A}) := \{\lambda \in \mathbb{Y} : P_\lambda(\mathbf{A} \mid q, t) > 0\} \quad (2.3)$$

is the *support* of a specialization \mathbf{A} .

There is no known classification of (q, t) -nonnegative specializations. However, a wide class of such specializations was introduced by Kerov [45, II.9], and he conjectured that they exhaust all Macdonald-nonnegative specializations (see also §2.4 below for more discussion).

These specializations depend on nonnegative parameters $\{\alpha_i\}_{i \geq 1}$, $\{\beta_i\}_{i \geq 1}$ and γ such that $\sum_{i=1}^{\infty} (\alpha_i + \beta_i) < \infty$. For definiteness, we will always assume that $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$ and $\beta_1 \geq \beta_2 \geq \dots \geq 0$. The corresponding specialization is defined on the power sums via the exponent of a generating function (in a formal variable u) as follows:

$$\exp \left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{1-t^n}{1-q^n} p_n(\mathbf{A}) u^n \right) = \exp(\gamma u) \prod_{i \geq 1} \frac{(t\alpha_i u; q)_{\infty}}{(\alpha_i u; q)_{\infty}} (1 + \beta_i u) =: \Pi(u; \mathbf{A}), \quad (2.4)$$

where the (infinite) q -Pochhammer symbol is defined as

$$(a; q)_{\infty} := \prod_{i=0}^{\infty} (1 - aq^i) = (1-a)(1-aq)(1-aq^2) \dots$$

In more detail, (2.4) means that

$$p_1(\mathbf{A}) = \sum_{i \geq 1} \alpha_i + \left(\gamma + \sum_{i \geq 1} \beta_i \right) \frac{1-q}{1-t}, \quad p_k(\mathbf{A}) = \sum_{i \geq 1} \alpha_i^k + (-1)^{k-1} \frac{1-q^k}{1-t^k} \sum_{i \geq 1} \beta_i^k, \quad (2.5)$$

where $k = 2, 3, \dots$. It can be verified that (2.4) defines (q, t) -nonnegative specializations, cf. [8, Prop. 2.2.2]. In the Hall–Littlewood case (i.e., when $q = 0$), the product in (2.4) turns into $\Pi(u; \mathbf{A}) = e^{\gamma u} \prod_{i \geq 1} \frac{1-t\alpha_i u}{1-\alpha_i u} (1 + \beta_i u)$.

Remark 2.5. When $\gamma = 0$, all $\beta_i = 0$, and only finitely many of the α_i 's are nonzero, then the specialization defined by (2.4) reduces to a finite length specialization (2.2).

In view of Remark 2.5, we will refer to the α_i 's as to the *usual variables*. We will also call the β_i 's the *dual variables* (the name is motivated by the presence of a certain duality involving transposition of Young diagrams, see §5.3 below).¹⁰ The parameter γ will be called the *Plancherel parameter*. We will denote by $(\boldsymbol{\alpha}; \boldsymbol{\beta}; \mathbf{Pl}_{\gamma})$ the specialization defined by (2.4) with parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots)$, and γ . We will always assume that the specialization $(\boldsymbol{\alpha}; \boldsymbol{\beta}; \mathbf{Pl}_{\gamma})$ is *nontrivial* (i.e., not all of the parameters are equal to zero). This property is equivalent to requiring that $p_1(\boldsymbol{\alpha}; \boldsymbol{\beta}; \mathbf{Pl}_{\gamma}) > 0$. If all α_i and all β_j are zero, we will call such a specialization a *pure Plancherel* specialization, and will denote it simply by \mathbf{Pl}_{γ} .

Clearly, a multiple $a \cdot (\boldsymbol{\alpha}; \boldsymbol{\beta}; \mathbf{Pl}_{\gamma})$ of the specialization $(\boldsymbol{\alpha}; \boldsymbol{\beta}; \mathbf{Pl}_{\gamma})$ corresponds to multiplying all the parameters α_i, β_i , and γ by this factor a . Union of specializations $(\boldsymbol{\alpha}; \boldsymbol{\beta}; \mathbf{Pl}_{\gamma}) \cup (\boldsymbol{\alpha}'; \boldsymbol{\beta}'; \mathbf{Pl}_{\gamma'})$ leads to the new parameters $\boldsymbol{\alpha} \cup \boldsymbol{\alpha}'$, $\boldsymbol{\beta} \cup \boldsymbol{\beta}'$ (these are unions as sets), and to the addition of the Plancherel parameters γ and γ' .

⁹Sometimes we will also use the term *Macdonald-nonnegative*, cf. [8, §2.2.1].

¹⁰Sometimes to emphasize that we are working with dual variables, we will use the hat notation. For example, a dual variable equal to 1 will be denoted by $\hat{1}$.

Remark 2.6. Note that our notation (borrowed from [8] and also used in [16]) differs from the one used by Kerov [45], see also, e.g., [37]. Namely, take a specialization $\mathbf{A} = (\boldsymbol{\alpha}; \boldsymbol{\beta}; \mathbf{Pl}_\gamma)$ described by (2.5), and consider other parameters $\tilde{\alpha}_i$, $\tilde{\beta}_j$ and $\tilde{\gamma}$ defined as

$$\tilde{\gamma} = \frac{1-q}{1-t}\gamma, \quad \{\tilde{\alpha}_r\}_{r=1}^\infty = \{\alpha_i\}_{i=1}^\infty \cup \{-q\beta_i t^{j-1}\}_{i,j=1}^\infty, \quad \{\tilde{\beta}_r\}_{r=1}^\infty = \{\beta_i t^{j-1}\}_{i,j=1}^\infty.$$

(In the Hall–Littlewood ($q = 0$) case, passing to these new parameters reduces to rescaling the Plancherel parameter and replacing each β_i by the geometric sequence $\beta_i, \beta_i t, \beta_i t^2, \dots$) Then the map

$$p_k \mapsto \tilde{\gamma} \mathbf{1}_{k=1} + \sum_{i \geq 1} \tilde{\alpha}_i^k + (-1)^{k-1} \sum_{i \geq 1} \tilde{\beta}_i^k, \quad k \geq 1 \quad (2.6)$$

is the same as (2.5). References [45] and [37] use parametrization (2.6) of specializations.

2.4. Remark: Completeness of the list of (q, t) -nonnegative specializations. The fact that specializations (2.4) indeed exhaust all possible (q, t) -nonnegative specializations was established in the following particular cases of parameters q and t :

- (1) $t = q^\theta$ and $q \rightarrow 1$, where $\theta > 0$ is a new parameter [46]. In this case the Macdonald symmetric functions reduce to the Jack symmetric functions introduced in [40], [41] (see also [50, VI.10]).
- (2) When $\theta = 1$ in (1), the Jack symmetric functions become the Schur symmetric functions. The statement about nonnegative specializations in this case is equivalent to the classification of totally nonnegative triangular Toeplitz matrices [23], [1], and to the classification of extreme characters of the infinite symmetric group [66]. See also [68], [69].
- (3) When $q = 0$ and $t = 1$, the Macdonald polynomials degenerate to the monomial symmetric functions. The classification of $(0, 1)$ -nonnegative specializations is equivalent to classification of partition structures in the sense of Kingman [47] (see also [43]). In this case, the parameters $\{\beta_i\}$ do not enter the classification. One can also view this as a particular case $\theta = 0$ in (1).
- (4) Another interesting particular case is $q = 0$ and $t = -1$, and the corresponding classification result is given in [52] (see also [39]). The $(0, -1)$ -nonnegative specializations are related to projective characters of the infinite symmetric group.

Cases (2), (3), and (4) above fall under the general Hall–Littlewood picture which corresponds to $q = 0$.¹¹ The classification result for $(0, t)$ -nonnegative specializations (we also refer to them as to *HL-nonnegative specializations*) has not been proven for general values of the parameter t .

When $t = q^{-1} = p^{-d} \in (0, 1)$ is the inverse of a prime power, the classification of HL-nonnegative specializations is related to random infinite triangular matrices over the finite field F_q , see §1.1.

3. COHERENT MEASURES ON PARTITIONS

¹¹However, in the present paper we restrict ourselves to $t \in [0, 1)$, which excludes cases (3) and (4) from the consideration.

3.1. (q, t) -coherent measures. For each fixed $n \geq 0$, consider a probability measure on partitions with n boxes defined as follows:

$$\mathcal{M}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}(\lambda) := \frac{n!}{(p_1(\alpha; \beta; \mathbf{Pl}_\gamma))^n} P_\lambda(\alpha; \beta; \mathbf{Pl}_\gamma | q, t) Q_\lambda(\mathbf{Pl}_1 | q, t), \quad \lambda \in \mathbb{Y}_n. \quad (3.1)$$

Here the first specialization $(\alpha; \beta; \mathbf{Pl}_\gamma)$ is any Macdonald nonnegative specialization defined by (2.4), the second specialization \mathbf{Pl}_1 is the pure Plancherel specialization with parameter $\gamma = 1$, and $p_1(\alpha; \beta; \mathbf{Pl}_\gamma)$ is given in (2.5). Note that $p_1(\alpha; \beta; \mathbf{Pl}_\gamma)$ also depends on (q, t) , but we omit this dependence.

Lemma 3.1. *Expression (3.1) indeed defines a probability measure on \mathbb{Y}_n , i.e.,*

$$\mathcal{M}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}(\lambda) \geq 0 \quad \text{for all } \lambda \in \mathbb{Y}_n, \quad \text{and} \quad \sum_{\lambda \in \mathbb{Y}_n} \mathcal{M}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}(\lambda) = 1.$$

Proof. The nonnegativity follows from the fact that $(\alpha; \beta; \mathbf{Pl}_\gamma)$ is a nonnegative specialization. To show that the weights sum to one, we use the identity

$$n! \sum_{\lambda \in \mathbb{Y}_n} P_\lambda(\mathbf{x} | q, t) Q_\lambda(\mathbf{Pl}_1 | q, t) = (p_1(\mathbf{x}))^n$$

which is a particular case of Lemma 5.8 below (corresponding to setting $\lambda = \emptyset$ in (5.20)), and take $\mathbf{x} = (\alpha; \beta; \mathbf{Pl}_\gamma)$. \square

We will call (3.1) the (q, t) -coherent measures (about the name, see §5.4). In the Hall–Littlewood case $q = 0$, we will refer to the $(0, t)$ -coherent measures as to the *HL-coherent measures*, and will denote them by $\mathcal{HL}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}$. The HL-coherent measures are the main object of the present paper, they are related to random infinite triangular matrices over a finite field, see §1.1, and (1.4) in particular.¹²

Remark 3.2. By the homogeneity of p_1 and P_λ in (3.1), the measure $\mathcal{M}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}$ is invariant under multiplication of the specialization $(\alpha; \beta; \mathbf{Pl}_\gamma)$ by any positive number. Thus, to simplify certain formulas below, we will sometimes assume that the specialization is such that $p_1(\alpha; \beta; \mathbf{Pl}_\gamma) = 1$.

3.2. Poissonization and Macdonald measures. Let $\tau > 0$ be a new parameter (later it will play the role of time), and let us *mix* the measures $\mathcal{M}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}$ by means of the Poisson distribution with the parameter $\tau p_1(\alpha; \beta; \mathbf{Pl}_\gamma)$ on the set of indices n :¹³

$$\mathcal{MM}_\tau^{\alpha; \beta; \mathbf{Pl}_\gamma} := e^{-\tau p_1(\alpha; \beta; \mathbf{Pl}_\gamma)} \sum_{n=0}^{\infty} \frac{(\tau p_1(\alpha; \beta; \mathbf{Pl}_\gamma))^n}{n!} \mathcal{M}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}. \quad (3.2)$$

That is, $\mathcal{MM}_\tau^{\alpha; \beta; \mathbf{Pl}_\gamma}$ is the probability measure on the set \mathbb{Y} of *all* Young diagrams, and from (3.1) we get

$$\mathcal{MM}_\tau^{\alpha; \beta; \mathbf{Pl}_\gamma}(\lambda) = e^{-\tau p_1(\alpha; \beta; \mathbf{Pl}_\gamma)} P_\lambda(\alpha; \beta; \mathbf{Pl}_\gamma | q, t) Q_\lambda(\mathbf{Pl}_\tau | q, t), \quad \lambda \in \mathbb{Y}. \quad (3.3)$$

¹²Note that because P_λ is a multiple of Q_λ , (3.1) reduces to (1.4) when $q = 0$, $t = q^{-1}$, and $p_1(\alpha; \beta; \mathbf{Pl}_\gamma) = 1$.

¹³The reason for the multiplication of τ by $p_1(\alpha; \beta; \mathbf{Pl}_\gamma)$ is the future convenience of certain formulas. Note that for now we are not assuming that $p_1(\alpha; \beta; \mathbf{Pl}_\gamma) = 1$ (see Remark 3.2).

The poissonized measures (3.3) belong to the class of *Macdonald measures* of [8], see also [31]. (This is why we use the notation \mathcal{MM} .) One can recover \mathcal{M}_n from \mathcal{MM}_τ by conditioning on the event that the Young diagram λ distributed according to \mathcal{MM}_τ has exactly n boxes.

Remark 3.3. The passage from \mathcal{MM}_τ to \mathcal{M}_n may be called *de-poissonization*. There are analytic tools relating poissonized and de-poissonized measures (e.g., see [3]), but for the purposes of the Law of Large Numbers (Theorem 1.4) we do not need to employ them.

We continue the discussion of Macdonald measures in §4.1 below.

4. MACDONALD PROCESSES AND BIVARIATE CONTINUOUS-TIME ‘DYNAMICS’

Here we recall and extend the general formalism of [16] for constructing *formal* continuous-time Markov jump ‘dynamics’ which map Macdonald processes to Macdonald processes (with evolved parameters). Formality means that we allow ‘dynamics’ to have negative ‘jump rates’ or ‘transition probabilities’ (we will indicate the absence of the positivity assumption with single quotation marks). All our results can be restated in linear algebraic terms, as statements about action of formal Markov semigroups (that is, we allow the presence of negative numbers in transition matrices) on probability measures. However, to make the discussion more understandable, we will use probabilistic language even when speaking about formal ‘dynamics’.

Remark 4.1. The sampling algorithm which we construct at the Hall–Littlewood ($q = 0$) level in §6 below involves only nonnegative probabilities.

4.1. Macdonald measures and Macdonald processes. Let \mathbf{A} be a (q, t) -nonnegative specialization of the algebra of symmetric functions (§2). Let $\tau \geq 0$ be a parameter. We will consider the following *Macdonald measures* on Young diagrams:

$$\mathcal{MM}_\tau^{\mathbf{A}}(\lambda) := \frac{P_\lambda(\mathbf{A} \mid q, t) Q_\lambda(\mathbf{P}\mathbf{I}_\tau \mid q, t)}{\Pi(\mathbf{A}; \mathbf{P}\mathbf{I}_\tau)}, \quad \lambda \in \mathbb{Y}. \quad (4.1)$$

Here for any two specializations \mathbf{A}, \mathbf{B} we have set

$$\Pi(\mathbf{A}; \mathbf{B}) := \exp \left(\sum_{n=0}^{\infty} \frac{1}{n} \frac{1-t^n}{1-q^n} p_n(\mathbf{A}) p_n(\mathbf{B}) \right) \quad (4.2)$$

provided that this expression is finite.¹⁴ The normalization of the measures (4.1) follows from the Cauchy identity [50, VI]

$$\sum_{\lambda \in \mathbb{Y}} P_\lambda(\mathbf{A} \mid q, t) Q_\lambda(\mathbf{B} \mid q, t) = \Pi(\mathbf{A}; \mathbf{B}). \quad (4.3)$$

Note that we consider only a particular case of the Macdonald measures when one of the specializations is a pure Plancherel specialization. In this case $\Pi(\mathbf{A}; \mathbf{P}\mathbf{I}_\tau) = e^{\tau p_1(\mathbf{A})} < \infty$ for any specialization \mathbf{A} , cf. §3.2. See also [8, §2], [9] about more general Macdonald measures.

Remark 4.2. For $\tau = 0$, the measure $\mathcal{MM}_\tau^{\mathbf{A}}$ (4.1) is concentrated on the empty diagram $\emptyset \in \mathbb{Y}$. For any $\tau > 0$, the support of this measure coincides with the support $\mathbb{Y}(\mathbf{A}) \subseteq \mathbb{Y}$ of the specialization \mathbf{A} (see (2.3)) because the value of $Q_\lambda(\mathbf{P}\mathbf{I}_\tau \mid q, t)$ is strictly positive for all $\lambda \in \mathbb{Y}$.

¹⁴ Note that the expression $\Pi(u; \mathbf{A})$ in (2.4) is a particular case of (4.2) corresponding to $\mathbf{B} = (u)$, a specialization into a single usual variable.

Let now \mathbf{A} and \mathbf{B} be two (q, t) -nonnegative specializations with $\Pi(\mathbf{A}; \mathbf{B}) < \infty$. There is a certain *stochastic link* mapping the measure $\mathcal{MM}_\tau^{\mathbf{A} \cup \mathbf{B}}$ to $\mathcal{MM}_\tau^{\mathbf{A}}$. (Here $\mathbf{A} \cup \mathbf{B}$ is the union of specializations, cf. §2.3.) Namely, consider the following matrices with rows and columns indexed by Young diagrams:

$$\Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}}(\lambda, \bar{\lambda}) := \frac{P_{\bar{\lambda}}(\mathbf{A})}{P_{\bar{\lambda}}(\mathbf{A} \cup \mathbf{B})} P_{\lambda/\bar{\lambda}}(\mathbf{B}), \quad \lambda \in \mathbb{Y}(\mathbf{A} \cup \mathbf{B}), \quad \bar{\lambda} \in \mathbb{Y}(\mathbf{A}). \quad (4.4)$$

Proposition 4.3 ([8, §2.3.1]). *The quantities $\Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}}(\lambda, \bar{\lambda})$ are nonnegative and*

$$\sum_{\bar{\lambda} \in \mathbb{Y}(\mathbf{A})} \Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}}(\lambda, \bar{\lambda}) = 1$$

(hence the name “stochastic link”). Moreover,

$$\mathcal{MM}_\tau^{\mathbf{A} \cup \mathbf{B}} \Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}} = \mathcal{MM}_\tau^{\mathbf{A}}. \quad (4.5)$$

The latter identity is understood in the matrix sense, and the measures $\mathcal{MM}_\tau^{\mathbf{A} \cup \mathbf{B}}$ and $\mathcal{MM}_\tau^{\mathbf{A}}$ should be viewed as row vectors.

One should understand (4.5) as a *compatibility relation* between the Macdonald measures $\mathcal{MM}_\tau^{\mathbf{A} \cup \mathbf{B}}$ on $\mathbb{Y}(\mathbf{A} \cup \mathbf{B})$ and $\mathcal{MM}_\tau^{\mathbf{A}}$ on $\mathbb{Y}(\mathbf{A})$. Now let us consider the *joint distribution* of the pair of Young diagrams $(\bar{\lambda}, \lambda)$ which arises from that relation. This joint distribution is supported on the subset¹⁵

$$\mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B}) := \left\{ \begin{bmatrix} \lambda \\ \bar{\lambda} \end{bmatrix} \in \mathbb{Y}(\mathbf{A}) \times \mathbb{Y}(\mathbf{A} \cup \mathbf{B}) : \Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}}(\lambda, \bar{\lambda}) > 0 \right\} \subseteq \mathbb{Y}(\mathbf{A}) \times \mathbb{Y}(\mathbf{A} \cup \mathbf{B}), \quad (4.6)$$

and is given by

$$\mathcal{MP}_\tau^{\mathbf{A}; \mathbf{B}}(\bar{\lambda}; \lambda) := \mathcal{MM}_\tau^{\mathbf{A} \cup \mathbf{B}}(\lambda) \Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}}(\lambda, \bar{\lambda}) = \frac{P_{\bar{\lambda}}(\mathbf{A}) P_{\lambda/\bar{\lambda}}(\mathbf{B}) Q_\lambda(\mathbf{Pl}_\tau)}{\Pi(\mathbf{A}; \mathbf{Pl}_\tau) \Pi(\mathbf{B}; \mathbf{Pl}_\tau)}, \quad \begin{bmatrix} \lambda \\ \bar{\lambda} \end{bmatrix} \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B}). \quad (4.7)$$

This distribution is a particular case of a *Macdonald process* introduced in [8, §2.2] (see also [9]).

If in the definition (4.7) one replaces the measure $\mathcal{MM}_\tau^{\mathbf{A} \cup \mathbf{B}}(\lambda)$ by *any* probability measure on $\mathbb{Y}(\mathbf{A} \cup \mathbf{B})$, then the resulting measure on $\mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$ will be *compatible* with $\Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}}$ in a way similar to (4.5). We will refer to this wider class of measures as to the *Gibbs measures*.

4.2. Univariate dynamics. Let us now describe certain continuous-time Markov jump dynamics on \mathbb{Y} which act nicely on Macdonald measures.¹⁶ Let \mathbf{A} be a (q, t) -nonnegative specialization. The *univariate* continuous-time Markov dynamics introduced in [8, §2.3.1] (see also [16, §4.3]) lives on the set of Young diagrams $\mathbb{Y}(\mathbf{A})$ and (during time $\sigma \geq 0$) maps the Macdonald measure $\mathcal{MM}_\tau^{\mathbf{A}}$ into the measure $\mathcal{MM}_{\tau+\sigma}^{\mathbf{A}}$ with evolved time parameter $\tau + \sigma$. This

¹⁵Note that the condition $\begin{bmatrix} \lambda \\ \bar{\lambda} \end{bmatrix} \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$ implies that $\bar{\lambda} \subseteq \lambda$.

¹⁶These dynamics may be viewed as discrete (q, t) -analogues of the classical Dyson Brownian motion [22] from random matrix theory, e.g., see [17].

univariate dynamics is defined through the jump rate matrix having the form

$$Q_{\mathbf{A}}(\lambda, \nu) := \begin{cases} \frac{P_{\nu}(\mathbf{A})}{P_{\lambda}(\mathbf{A})} \psi'_{\nu/\lambda}, & \text{if } \lambda \nearrow \nu; \\ - \sum_{\square \in \mathcal{U}(\lambda)} Q_{\mathbf{A}}(\lambda, \lambda + \square), & \text{if } \nu = \lambda; \\ 0, & \text{otherwise.} \end{cases} \quad (4.8)$$

Here

$$\psi'_{\nu/\lambda} = \psi'_{\nu/\lambda}(q, t) := Q_{\nu/\lambda}(\hat{1} \mid q, t) \quad (4.9)$$

is the value of the skew Macdonald symmetric function under the specialization into one dual variable (equal to one), it is given by [50, VI.(6.24.iv)], see also §5.1 below.

We summarize properties of the univariate dynamics in the following proposition:

Proposition 4.4. (1) *Jump rates $Q_{\mathbf{A}}$ define a Feller Markov jump process with semigroup $\{P_{\mathbf{A}}(\tau)\}_{\tau \geq 0}$, where $P_{\mathbf{A}}(\tau) = \exp(\tau Q_{\mathbf{A}})$.*

(2) *The action of the univariate dynamics on Macdonald measures is given by*

$$\mathcal{M}\mathcal{M}_{\tau}^{\mathbf{A}} P_{\mathbf{A}}(\sigma) = \mathcal{M}\mathcal{M}_{\tau+\sigma}^{\mathbf{A}}, \quad \sigma \geq 0. \quad (4.10)$$

(3) *The univariate dynamics are compatible with the stochastic links in the sense that (as $\mathbb{Y}(\mathbf{A} \cup \mathbf{B}) \times \mathbb{Y}(\mathbf{A})$ matrices)*

$$\Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}} Q_{\mathbf{A}} = Q_{\mathbf{A} \cup \mathbf{B}} \Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}} \quad \text{and} \quad \Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}} P_{\mathbf{A}}(\tau) = P_{\mathbf{A} \cup \mathbf{B}}(\tau) \Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}}, \quad \tau \geq 0. \quad (4.11)$$

In other words, “the following diagram is commutative”:

$$\begin{array}{ccc} \mathbb{Y}(\mathbf{A} \cup \mathbf{B}) & \xrightarrow{P_{\mathbf{A} \cup \mathbf{B}}(\tau)} & \mathbb{Y}(\mathbf{A} \cup \mathbf{B}) \\ \downarrow \Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}} & & \downarrow \Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}} \\ \mathbb{Y}(\mathbf{A}) & \xrightarrow{P_{\mathbf{A}}(\tau)} & \mathbb{Y}(\mathbf{A}) \end{array}$$

Proof. See [8, §2.3.1] and [16, §4.3]. □

4.3. Infinitesimal skew Cauchy identity. The skew Cauchy identity [50, VI.7] states that for two Macdonald nonnegative specializations \mathbf{A} and \mathbf{B} with $\Pi(\mathbf{A}; \mathbf{B}) < \infty$ one has

$$\sum_{\kappa \in \mathbb{Y}} P_{\kappa/\lambda}(\mathbf{A}) Q_{\kappa/\nu}(\mathbf{B}) = \Pi(\mathbf{A}; \mathbf{B}) \sum_{\mu \in \mathbb{Y}} Q_{\lambda/\mu}(\mathbf{B}) P_{\nu/\mu}(\mathbf{A}) \quad (4.12)$$

for any $\lambda, \nu \in \mathbb{Y}$. When $\lambda = \nu = \emptyset$, this identity turns into the usual Cauchy identity (4.3). We will need the following infinitesimal version of (4.12):

Proposition 4.5 (infinitesimal skew Cauchy identity). *Let $\mathbf{B} = \hat{\varepsilon}$ be the specialization into one dual variable equal to ε . Taking the coefficient by ε in both sides of (4.12) yields the following identity for any $\lambda, \nu \in \mathbb{Y}$:*

$$\sum_{\square \in \mathcal{U}(\nu)} P_{\nu+\square/\lambda}(\mathbf{A}) \psi'_{\nu+\square/\nu} = p_1(\mathbf{A}) P_{\nu/\lambda}(\mathbf{A}) + \sum_{\square \in \mathcal{D}(\lambda)} P_{\nu/\lambda-\square}(\mathbf{A}) \psi'_{\lambda/\lambda-\square}. \quad (4.13)$$

Proof. We have (see (2.5) and (4.2))

$$\Pi(\mathbf{A}; \hat{\varepsilon}) = \exp \left(\sum_{n \geq 0} p_n(\mathbf{A}) (-1)^{n-1} \varepsilon^n \right) = 1 + \varepsilon p_1(\mathbf{A}) + O(\varepsilon^2).$$

Now (4.12) takes the form

$$\sum_{\kappa \in \mathbb{Y}} P_{\kappa/\lambda}(\mathbf{A}) \varepsilon^{|\kappa| - |\nu|} \psi'_{\kappa/\nu} = (1 + \varepsilon p_1(\mathbf{A}) + O(\varepsilon^2)) \sum_{\mu \in \mathbb{Y}} \psi'_{\lambda/\mu} \varepsilon^{|\lambda| - |\mu|} (\mathbf{B}) P_{\nu/\mu}(\mathbf{A}).$$

The desired claim follows by considering the coefficient by ε in the above identity. \square

One can readily see that (4.13) is essentially equivalent to the commutation relation (4.11) between $\Lambda_{\mathbf{A}}^{\mathbf{A} \cup \mathbf{B}}$ and the univariate jump rate matrices (4.8). See [16, §2.4] for more detail.

Identity (4.13) readily implies that the diagonal elements of the jump rate matrix (4.8) are

$$\mathbf{Q}_{\mathbf{A}}(\nu, \nu) = -p_1(\mathbf{A}). \quad (4.14)$$

Indeed, one needs to put $\lambda = \emptyset$ in (4.13) (this kills the sum in the right-hand side), and divide both sides by $P_{\nu/\lambda}(\mathbf{A}) = P_{\nu}(\mathbf{A})$.

4.4. Bivariate ‘dynamics’. This subsection extends results of [16, §2 and §5]. See also [21], [10], and [13, §8] for related constructions, and [17] for a survey.

We call a continuous-time Markov ‘dynamics’¹⁷ on the space $\mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$ (4.6) with matrix of ‘jump rates’ $\mathbf{Q}_{\mathbf{A}; \mathbf{B}}^{(2)}$ a *bivariate ‘dynamics’* if the following three conditions are satisfied:

- (1) The ‘dynamics’ $\mathbf{Q}_{\mathbf{A}; \mathbf{B}}^{(2)}$ preserves the class of Gibbs measures on $\mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$.
- (2) Assume that $\mathbf{Q}_{\mathbf{A}; \mathbf{B}}^{(2)}$ starts from a Gibbs measure on $\mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$. Then on the *upper level* $\mathbb{Y}(\mathbf{A} \cup \mathbf{B})$, the ‘dynamics’ $\mathbf{Q}_{\mathbf{A}; \mathbf{B}}^{(2)}$ must reduce to the univariate dynamics $\mathbf{Q}_{\mathbf{A} \cup \mathbf{B}}$.
- (3) The ‘dynamics’ $\mathbf{Q}_{\mathbf{A}; \mathbf{B}}^{(2)}$ evolves according to a *sequential update*, with interaction propagating from the lower to the upper level. Note that by Proposition 4.4.(3), sequential update property plus the above condition (2) imply that on the lower level $\mathbb{Y}(\mathbf{A})$, $\mathbf{Q}_{\mathbf{A}; \mathbf{B}}^{(2)}$ must reduce to the corresponding univariate dynamics $\mathbf{Q}_{\mathbf{A}}$. Hence, the ‘jump rates’ of the bivariate ‘dynamics’ must have the form (here $[\frac{\lambda}{\bar{\lambda}}], [\frac{\nu}{\bar{\nu}}] \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$):

$$\mathbf{Q}_{\mathbf{A}; \mathbf{B}}^{(2)}([\frac{\lambda}{\bar{\lambda}}], [\frac{\nu}{\bar{\nu}}]) = \begin{cases} W(\lambda, \nu | \bar{\nu}), & \text{if } \bar{\lambda} = \bar{\nu}; \\ \mathbf{Q}_{\mathbf{A}}(\bar{\lambda}, \bar{\nu}) V(\lambda, \nu | \bar{\lambda}, \bar{\nu}), & \text{if } \bar{\lambda} \neq \bar{\nu}; \\ \mathbf{Q}_{\mathbf{A}}(\bar{\nu}, \bar{\nu}) + W(\nu, \nu | \bar{\nu}), & \text{if } \bar{\lambda} = \bar{\nu} \text{ and } \lambda = \nu. \end{cases} \quad (4.15)$$

Here $W(\lambda, \nu | \bar{\nu})$ is the ‘rate’ of an independent jump $\lambda \rightarrow \nu$ on the upper level (given that there were no jumps on the lower level, so $\bar{\nu} = \bar{\lambda}$). We assume that these ‘rates’ satisfy

$$\sum_{\nu \neq \lambda} W(\lambda, \nu | \bar{\nu}) = -W(\lambda, \lambda | \bar{\nu}) \quad \text{for all } [\frac{\lambda}{\bar{\lambda}}] \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B}). \quad (4.16)$$

The quantity $V(\lambda, \nu | \bar{\lambda}, \bar{\nu})$ is the ‘conditional probability’ that the jump $\bar{\lambda} \rightarrow \bar{\nu}$ on the lower level triggers an instantaneous move $\lambda \rightarrow \nu$ on the upper level. Note that we do not forbid

¹⁷This actually is the first place when we drop the nonnegativity assumption.

the possibility that $\lambda = \nu$, i.e., that the jump does not propagate upwards (we will soon forbid such moves, see §4.5 below). The ‘probabilities’ of triggered moves must satisfy

$$V(\lambda, \nu | \bar{\lambda}, \bar{\lambda}) = \mathbf{1}_{\lambda=\nu}, \quad \sum_{\nu} V(\lambda, \nu | \bar{\lambda}, \bar{\nu}) = 1 \quad (4.17)$$

(where $[\frac{\lambda}{\bar{\lambda}}], [\frac{\nu}{\bar{\lambda}}] \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$ and $[\frac{\lambda}{\bar{\lambda}}], [\frac{\nu}{\bar{\nu}}] \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$ in the first and in the second equality, respectively).

Note that by (4.8), (4.16), and (4.17),

$$\sum_{[\frac{\lambda}{\bar{\lambda}}] \neq [\frac{\nu}{\bar{\nu}}]} Q_{\mathbf{A}; \mathbf{B}}^{(2)}([\frac{\lambda}{\bar{\lambda}}], [\frac{\nu}{\bar{\nu}}]) = -Q_{\mathbf{A}; \mathbf{B}}^{(2)}([\frac{\lambda}{\bar{\lambda}}], [\frac{\lambda}{\bar{\lambda}}]),$$

as it should be for a matrix of ‘jump rates’.

We see that bivariate ‘dynamics’ describe ways to *stitch together* the univariate dynamics $Q_{\mathbf{A}}$ and $Q_{\mathbf{A} \cup \mathbf{B}}$ into a Markov ‘dynamics’ on the space $\mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$. Such a stitching is not unique, and all possible ways to construct a bivariate ‘dynamics’ can be characterized as follows:

Theorem 4.6 ([16]). *The ‘jump rates’ W and the ‘probabilities’ of triggered moves V satisfying (4.16) and (4.17) correspond to a bivariate ‘dynamics’ if and only if*

$$\begin{aligned} \sum_{\bar{\square} \in \mathcal{D}(\bar{\nu})} V(\lambda, \lambda + \square | \bar{\nu} - \bar{\square}, \bar{\nu}) P_{\lambda/\bar{\nu}-\bar{\square}}(\mathbf{B}) \psi'_{\bar{\nu}/\bar{\nu}-\bar{\square}} \\ + W(\lambda, \lambda + \square | \bar{\nu}) P_{\lambda/\bar{\nu}}(\mathbf{B}) = P_{\lambda+\square/\bar{\nu}}(\mathbf{B}) \psi'_{\lambda+\square/\lambda}, \end{aligned} \quad (4.18)$$

for all $\bar{\nu} \in \mathbb{Y}(\mathbf{A})$, $\lambda \in \mathbb{Y}(\mathbf{A} \cup \mathbf{B})$, and all $\square \in \mathcal{U}(\lambda)$, such that $[\frac{\lambda+\square}{\bar{\nu}}] \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$.

Idea of proof. This is established in the same way as [16, Prop. 5.3] with the help of the general discussion of [16, §2.4]. The idea of the proof is the following. Fix $\lambda \in \mathbb{Y}(\mathbf{A} \cup \mathbf{B})$ and sample $\bar{\lambda} \in \mathbb{Y}(\mathbf{A})$ according to the Gibbs property. The new configuration $[\frac{\nu}{\bar{\nu}}]$ (arising after an infinitesimal amount of time) can be reached in two ways: either by running the bivariate dynamics $Q_{\mathbf{A}; \mathbf{B}}^{(2)}$ (4.15), or by running the univariate dynamics $Q_{\mathbf{A} \cup \mathbf{B}}$ to get from λ to ν , and then sampling $\bar{\nu}$ according to the Gibbs property. Thus, we arrive at two different expressions for the infinitesimal probability of the resulting configuration $[\frac{\nu}{\bar{\nu}}]$, and (4.18) is the equality between them. In this equality the Young diagrams $\bar{\nu} = \lambda + \square$, ν , and λ are fixed, and the summation goes over $\bar{\lambda} = \bar{\nu} - \bar{\square}$. \square

Remark 4.7. Let us make several comments about the general identity (4.18):

- (1) Due to properties (1)–(2) of the bivariate ‘dynamics’, during an infinitesimally small time interval under $Q_{\mathbf{A}; \mathbf{B}}^{(2)}$, at most one box can be added to each of the Young diagrams on the lower and on the upper level. In (4.18), these added boxes are denoted by $\bar{\square}$ and \square , respectively.
- (2) Identity (4.18) is written down for each fixed new state $\bar{\nu}$ on the lower level and all possible moves $\lambda \rightarrow \lambda + \square$ on the upper level. The summation in the left-hand side is over all “histories” $\bar{\nu} - \bar{\square} \rightarrow \bar{\nu}$ on the lower level.
- (3) Observe that (4.18) is essentially independent of the lower specialization \mathbf{A} . That is, it depends on \mathbf{A} only through the requirement that $\bar{\nu} \in \mathbb{Y}(\mathbf{A})$, so \mathbf{A} does not affect the form of the identity (4.18) which is written down for each fixed $\bar{\nu}$ separately.

- (4) If we sum (4.18) over all $\square \in \mathcal{U}(\lambda)$, then we get the infinitesimal skew Cauchy identity (4.13). Thus, one may think that bivariate ‘dynamics’ correspond to refinements of (4.13) (or, equivalently, of the commutation relations (4.11)).
- (5) Identity (4.18) extends the results of [16] in the sense that the latter paper deals only with the case when \mathbf{B} is a specialization into a single usual variable. For such \mathbf{B} , the Young diagrams $\bar{\nu}$ and $\lambda + \square$ in (4.18) must differ by a horizontal strip (and hence the pair $\left[\begin{smallmatrix} \lambda + \square \\ \bar{\nu} \end{smallmatrix}\right]$ can be represented as a configuration of interlacing particles on two levels, cf. §2.1 and also §5.2 below). For other \mathbf{B} ’s, condition $\left[\begin{smallmatrix} \lambda + \square \\ \bar{\nu} \end{smallmatrix}\right] \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$ will be different (and can be more complicated).

4.5. RSK-type ‘dynamics’. In the present paper we will deal only with the following subclass of bivariate ‘dynamics’:

Definition 4.8. A bivariate ‘dynamics’ $\mathbf{Q}_{\mathbf{A}; \mathbf{B}}^{(2)}$ is called *RSK-type* if for any $\left[\begin{smallmatrix} \lambda \\ \bar{\nu} \end{smallmatrix}\right] \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$ and any $\bar{\square} \in \mathcal{D}(\bar{\nu})$, one has

$$V(\lambda, \lambda \mid \bar{\nu} - \bar{\square}, \bar{\nu}) = 0.$$

This means that a jump on the lower level always propagates to the upper level. See §6 below and also [16] for more discussion including connections to the classical RSK (Robinson–Schensted–Knuth) insertion algorithm.

Proposition 4.9. *For an RSK-type bivariate ‘dynamics’, the diagonal elements of the ‘jump rate’ matrix are given by*

$$\mathbf{Q}_{\mathbf{A}; \mathbf{B}}^{(2)}\left(\left[\begin{smallmatrix} \lambda \\ \bar{\lambda} \end{smallmatrix}\right], \left[\begin{smallmatrix} \lambda \\ \bar{\lambda} \end{smallmatrix}\right]\right) = \mathbf{Q}_{\mathbf{A} \cup \mathbf{B}}(\lambda, \lambda), \quad \left[\begin{smallmatrix} \lambda \\ \bar{\lambda} \end{smallmatrix}\right] \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B}).$$

Using (4.14), we also have $\mathbf{Q}_{\mathbf{A}; \mathbf{B}}^{(2)}\left(\left[\begin{smallmatrix} \lambda \\ \bar{\lambda} \end{smallmatrix}\right], \left[\begin{smallmatrix} \lambda \\ \bar{\lambda} \end{smallmatrix}\right]\right) = -p_1(\mathbf{A} \cup \mathbf{B})$, and $W(\lambda, \lambda \mid \bar{\lambda}) = -p_1(\mathbf{B})$.

Proof. Follows from [16, (2.20)] with $x_k = y_k = \lambda$ and $y_{k-1} = \bar{\lambda}$. By the definition of RSK-type ‘dynamics’, the sum over x_{k-1} in that identity reduces to only one summand corresponding to $x_{k-1} = y_{k-1}$. This leads to $\mathbf{Q}_{\mathbf{A}}(\bar{\lambda}, \bar{\lambda}) + W(\lambda, \lambda \mid \bar{\lambda}) = \mathbf{Q}_{\mathbf{A} \cup \mathbf{B}}(\lambda, \lambda)$, which is equivalent to the desired claim (see (4.15)). \square

5. THREE PARTICULAR BIVARIATE ‘DYNAMICS’ ON MACDONALD PROCESSES

Here we present three explicit examples of RSK-type bivariate ‘dynamics’ $\mathbf{Q}_{\mathbf{A}; (\alpha)}^{(2)}$, $\mathbf{Q}_{\mathbf{A}; (\beta)}^{(2)}$, and $\mathbf{Q}_{\mathbf{A}; \mathbf{PL}_\gamma}^{(2)}$, (where \mathbf{A} is an arbitrary (q, t) -nonnegative specialization, and $\alpha, \beta, \gamma > 0$). In §6 below we will use them as building blocks for our RSK-type sampling algorithm.

5.1. (q, t) -quantities and their properties. Let us write down explicit formulas for various quantities related to Macdonald polynomials, and also list some relations between them. Let us define $\psi_{\lambda/\mu}(q, t)$, $\varphi_{\lambda/\mu}(q, t)$, $\psi'_{\lambda/\mu}(q, t)$, and $\varphi'_{\lambda/\mu}(q, t)$ by the following one-variable specialization formulas for any $\lambda, \mu \in \mathbb{Y}$:

$$\begin{aligned} P_{\lambda/\mu}(\alpha \mid q, t) &= \alpha^{|\lambda| - |\mu|} \psi_{\lambda/\mu}(q, t) \mathbf{1}_{\mu \prec_h \lambda}; \\ Q_{\lambda/\mu}(\alpha \mid q, t) &= \alpha^{|\lambda| - |\mu|} \varphi_{\lambda/\mu}(q, t) \mathbf{1}_{\mu \prec_h \lambda}; \\ P_{\lambda/\mu}(\hat{\beta} \mid q, t) &= \beta^{|\lambda| - |\mu|} \varphi'_{\lambda/\mu}(q, t) \mathbf{1}_{\mu \prec_v \lambda}; \\ Q_{\lambda/\mu}(\hat{\beta} \mid q, t) &= \beta^{|\lambda| - |\mu|} \psi'_{\lambda/\mu}(q, t) \mathbf{1}_{\mu \prec_v \lambda} \end{aligned} \tag{5.1}$$

(recall that the notation $\hat{\beta}$ emphasizes that we are specializing polynomials into one dual variable). All formulas in the rest of this subsection describe various properties of the above quantities. They follow from properties of Macdonald symmetric functions, and we refer to [50, VI] for details.

Let us first list properties which do not involve swapping the parameters q and t . Define

$$b_\lambda(q, t) := \frac{1}{\langle P_\lambda(\cdot | q, t), P_\lambda(\cdot | q, t) \rangle_{q, t}} = \frac{Q_\lambda(\cdot | q, t)}{P_\lambda(\cdot | q, t)}, \quad \lambda \in \mathbb{Y},$$

see §2.2. (The dot means that we take symmetric functions in arbitrary variables.) An explicit formula for $b_\lambda(q, t)$ may be found in [50, VI.(6.19)], but we do not need it. We have

$$Q_{\lambda/\mu}(\cdot | q, t) = \frac{b_\lambda(q, t)}{b_\mu(q, t)} P_{\lambda/\mu}(\cdot | q, t); \quad (5.2)$$

$$\varphi_{\lambda/\mu}(q, t) = \frac{b_\lambda(q, t)}{b_\mu(q, t)} \psi_{\lambda/\mu}(q, t); \quad \varphi'_{\lambda/\mu}(q, t) = \frac{b_\mu(q, t)}{b_\lambda(q, t)} \psi'_{\lambda/\mu}(q, t); \quad (5.3)$$

In the special case when $\mu \nearrow \lambda$, one also has

$$\varphi_{\lambda/\mu}(q, t) = \frac{1-t}{1-q} \psi'_{\lambda/\mu}(q, t), \quad \varphi'_{\lambda/\mu}(q, t) = \frac{1-q}{1-t} \psi_{\lambda/\mu}(q, t). \quad (5.4)$$

Now let us turn to formulas involving swapping of q with t :

$$b_\lambda(q, t) = \frac{1}{b_{\lambda'}(t, q)}; \quad \psi'_{\lambda/\mu}(q, t) = \psi_{\lambda'/\mu'}(t, q); \quad \varphi'_{\lambda/\mu}(q, t) = \varphi_{\lambda'/\mu'}(t, q); \quad (5.5)$$

$$P_{\lambda/\mu}(\hat{\beta} | q, t) = Q_{\lambda'/\mu'}(\beta | t, q). \quad (5.6)$$

Remark 5.1. The “symmetry” between the parameters q and t in (5.5)–(5.6) follows from the existence of an endomorphism of \mathbf{Sym} defined by its action on the power sums as

$$\omega_{q,t}: p_k \mapsto (-1)^{k-1} \frac{1-q^k}{1-t^k} p_k, \quad k = 1, 2, \dots$$

We have $\omega_{q,t}\omega_{t,q} = \text{id}$, and

$$\omega_{q,t} P_{\lambda/\mu}(\mathbf{x} | q, t) = Q_{\lambda'/\mu'}(\mathbf{x} | t, q), \quad \omega_{q,t} Q_{\lambda/\mu}(\mathbf{x} | q, t) = P_{\lambda'/\mu'}(\mathbf{x} | t, q), \quad \lambda, \mu \in \mathbb{Y}.$$

One readily sees that applying the endomorphism $\omega_{t,q}: \mathbf{Sym} \rightarrow \mathbf{Sym}$ and then a (q, t) -nonnegative specialization $(\alpha; \beta; \mathbf{Pl}_\gamma | q, t)$ mapping \mathbf{Sym} to \mathbb{R} , one gets another specialization which is now (t, q) -nonnegative (note the swapping of the usual and dual variables):

$$(\alpha; \beta; \mathbf{Pl}_\gamma | q, t) \circ \omega_{t,q} = (\beta; \alpha; \mathbf{Pl}_{\frac{1-q}{1-t}\gamma} | t, q).$$

Finally, let us list several explicit formulas for the above (q, t) -quantities which we will use. For $\mu \prec_h \lambda$,

$$\psi_{\lambda/\mu}(q, t) = \prod_{1 \leq i \leq j \leq \ell(\mu)} \frac{f(q^{\mu_i - \mu_j} t^{j-i}) f(q^{\lambda_i - \lambda_{j+1}} t^{j-i})}{f(q^{\lambda_i - \mu_j} t^{j-i}) f(q^{\mu_i - \lambda_{j+1}} t^{j-i})}, \quad f(u) := \frac{(tu; q)_\infty}{(qu; q)_\infty}. \quad (5.7)$$

Let $\lambda = \mu + \square$ for some box $\square \in \mathcal{U}(\mu)$, and j be the row number of that box. We will denote this situation as $\lambda = \mu + \mathbf{e}_j$. In this case,

$$\psi'_{\mu+\mathbf{e}_j/\mu}(q, t) = \prod_{i=1}^{j-1} \frac{(1 - q^{\mu_i - \mu_j} t^{j-i-1})(1 - q^{\lambda_i - \lambda_j} t^{j-i+1})}{(1 - q^{\mu_i - \mu_j} t^{j-i})(1 - q^{\lambda_i - \lambda_j} t^{j-i})}. \quad (5.8)$$

We will also need certain combinations of the quantities $\psi_{\lambda/\mu}$ and $\psi'_{\mu+\square/\mu}$ which were employed in [16] (in particular, see [16, §5.4]). Namely, let $\bar{\nu}, \lambda \in \mathbb{Y}$ be such that $\bar{\nu} \prec_h \lambda$. Let a box $\bar{\square} \in \mathcal{D}(\bar{\nu})$ belong to row number i (so we can use the notation $\bar{\nu} - \bar{\square} = \bar{\nu} - \bar{\mathbf{e}}_i$). Define

$$\begin{aligned} T_i(\bar{\nu}, \lambda | q, t) &:= \frac{\psi_{\lambda/\bar{\nu}-\bar{\mathbf{e}}_i}(q, t)}{\psi_{\lambda/\bar{\nu}}(q, t)} \psi'_{\bar{\nu}/\bar{\nu}-\bar{\mathbf{e}}_i}(q, t) = \frac{(1 - q^{\lambda_i - \bar{\nu}_i} t)(1 - q^{\bar{\nu}_i - \lambda_{i+1}})}{(1 - q^{\lambda_i - \bar{\nu}_i + 1})(1 - q^{\bar{\nu}_i - 1 - \lambda_{i+1}} t)} \\ &\times \prod_{r=1}^{i-1} \frac{(1 - q^{\lambda_r - \bar{\nu}_i} t^{i-r+1})(1 - q^{\bar{\nu}_r - \bar{\nu}_i + 1} t^{i-r-1})}{(1 - q^{\lambda_r - \bar{\nu}_i + 1} t^{i-r})(1 - q^{\bar{\nu}_r - \bar{\nu}_i} t^{i-r})} \prod_{s=i+1}^{\ell(\bar{\nu})} \frac{(1 - q^{\bar{\nu}_i - \bar{\nu}_s - 1} t^{s-i+1})(1 - q^{\bar{\nu}_i - \lambda_{s+1}} t^{s-i})}{(1 - q^{\bar{\nu}_i - \bar{\nu}_s} t^{s-i})(1 - q^{\bar{\nu}_i - \lambda_{s+1} - 1} t^{s-i+1})}. \end{aligned} \quad (5.9)$$

Also, for $\bar{\nu} \prec_h \lambda$, let a box $\square \in \mathcal{U}(\lambda)$ belong to row number j . Define

$$\begin{aligned} S_j(\bar{\nu}, \lambda | q, t) &:= \frac{\psi_{\lambda+\mathbf{e}_j/\bar{\nu}}(q, t)}{\psi_{\lambda/\bar{\nu}}(q, t)} \psi'_{\lambda+\mathbf{e}_j/\lambda}(q, t) = \prod_{r=1}^{j-1} \frac{(1 - q^{\bar{\nu}_r - \lambda_j} t^{j-r-1})(1 - q^{\lambda_r - \lambda_j - 1} t^{j-r+1})}{(1 - q^{\bar{\nu}_r - \lambda_j - 1} t^{j-r})(1 - q^{\lambda_r - \lambda_j} t^{j-r})} \\ &\times \prod_{s=j}^{\ell(\bar{\nu})} \frac{(1 - q^{\lambda_j - \lambda_{s+1} + 1} t^{s-j})(1 - q^{\lambda_j - \bar{\nu}_s} t^{s-j+1})}{(1 - q^{\lambda_j - \lambda_{s+1}} t^{s-j+1})(1 - q^{\lambda_j - \bar{\nu}_s + 1} t^{s-j})}. \end{aligned} \quad (5.10)$$

The product formulas in (5.9)–(5.10) readily follow from (5.7) and (5.8).

5.2. Bivariate ‘dynamics’ with a usual variable. In this subsection we present functions $(W_{(\alpha)}, V_{(\alpha)})$ corresponding (as in §4.4) to an RSK-type bivariate ‘dynamics’ $\mathbf{Q}_{\mathbf{A};(\alpha)}^{(2)}$, where \mathbf{A} is an arbitrary (q, t) -nonnegative specialization and $\alpha > 0$.

Lemma 5.2. *The state space (4.6) of a bivariate ‘dynamics’ $\mathbf{Q}_{\mathbf{A};(\alpha)}^{(2)}$ can be characterized as*

$$\mathbb{Y}^{(2)}(\mathbf{A}; (\alpha)) = \left\{ \left[\frac{\lambda}{\bar{\lambda}} \right] : \bar{\lambda} \in \mathbb{Y}(\mathbf{A}) \text{ and } \bar{\lambda} \prec_h \lambda \right\}. \quad (5.11)$$

Proof. Clearly, the conditions $P_{\bar{\lambda}}(\mathbf{A} | q, t) > 0$ and $P_{\lambda/\bar{\lambda}}(\alpha | q, t) > 0$ are satisfied under conditions (5.11). Using identity [50, VI.(7.9’)] which can be written in the form

$$\sum_{\bar{\lambda}: \bar{\lambda} \prec_h \lambda} P_{\lambda/\bar{\lambda}}(\alpha | q, t) P_{\bar{\lambda}}(\mathbf{A} | q, t) = P_{\lambda}(\mathbf{A} \cup (\alpha) | q, t), \quad (5.12)$$

we conclude that the condition $P_{\lambda}(\mathbf{A} \cup (\alpha) | q, t) > 0$ also holds in (5.11). \square

Let us fix $h \in \{1, 2, \dots\} \cup \{+\infty\}$. Our functions $(W_{(\alpha)}, V_{(\alpha)}) = (W_{(\alpha)}^h, V_{(\alpha)}^h)$ will depend on h as a parameter.

We will describe the values $W_{(\alpha)}^h(\lambda, \cdot | \bar{\nu})$ and $V_{(\alpha)}^h(\lambda, \cdot | \cdot, \bar{\nu})$ for all meaningful pairs of Young diagrams $\bar{\nu} \in \mathbb{Y}(\mathbf{A})$ and $\lambda \in \mathbb{Y}(\mathbf{A} \cup \{\alpha\})$ (entering (4.16), (4.17), and (4.18)). Looking at

(4.15), we see that these two diagrams must satisfy $[\frac{\lambda+\square}{\bar{\nu}}] \in \mathbb{Y}^{(2)}(\mathbf{A}; \{\alpha\})$ for at least one box $\square \in \mathcal{U}(\lambda)$.

If $[\frac{\lambda}{\bar{\nu}}] \notin \mathbb{Y}^{(2)}(\mathbf{A}; \{\alpha\})$, then it means that there is a unique $\bar{\square} \in \mathcal{D}(\bar{\nu})$ such that $[\frac{\lambda}{\bar{\nu}-\bar{\square}}] \in \mathbb{Y}^{(2)}(\mathbf{A}; \{\alpha\})$. Pairs $[\frac{\lambda}{\bar{\nu}}] \notin \mathbb{Y}^{(2)}(\mathbf{A}; \{\alpha\})$ do not enter the definition of $W_{(\alpha)}^h$, and the only nonzero value of $V_{(\alpha)}^h$ in this case is

$$V_{(\alpha)}^h(\lambda, \lambda + \square \mid \bar{\nu} - \bar{\square}, \bar{\nu}) = 1, \quad [\frac{\lambda}{\bar{\nu}}] \notin \mathbb{Y}^{(2)}(\mathbf{A}; \{\alpha\}), \quad [\frac{\lambda+\square}{\bar{\nu}}], [\frac{\lambda}{\bar{\nu}-\bar{\square}}] \in \mathbb{Y}^{(2)}(\mathbf{A}; \{\alpha\}). \quad (5.13)$$

In words, this means that if a jump $\bar{\nu} - \bar{\square} \rightarrow \bar{\nu}$ on the lower level breaks the condition $[\frac{\lambda}{\bar{\nu}}] \in \mathbb{Y}^{(2)}(\mathbf{A}; \{\alpha\})$, then there is a unique jump $\lambda \rightarrow \lambda + \square$ which must happen (almost surely) to restore this condition. In [16, §5.3] the property (5.13) was called the *short-range pushing*.

In view of (5.13), it remains to define $W_{(\alpha)}^h(\lambda, \cdot \mid \bar{\nu})$ and $V_{(\alpha)}^h(\lambda, \cdot \mid \cdot, \bar{\nu})$ for all possible pairs $[\frac{\lambda}{\bar{\nu}}] \in \mathbb{Y}^{(2)}(\mathbf{A}; \{\alpha\})$. Denote $\ell := \ell(\bar{\nu})$, so $\ell(\lambda) \leq \ell + 1$. We represent pairs $[\frac{\lambda}{\bar{\nu}}]$ as interlacing particle configurations $\{\bar{\nu}_1, \dots, \bar{\nu}_\ell\} \sqcup \{\lambda_1, \dots, \lambda_{\ell+1}\} \subset \mathbb{Z} \sqcup \mathbb{Z}$ with ℓ and $\ell + 1$ particles on the lower and on the upper levels, respectively (appending λ by zeroes if necessary). See Fig. 5.

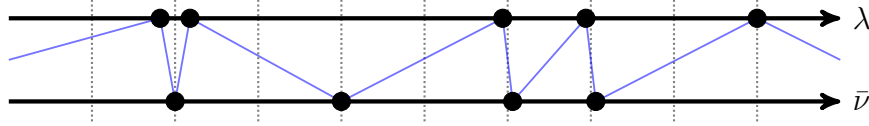


FIGURE 5. An interlacing particle configuration (zigzag illustrates the interlacing).

Clearly, adding a box to a Young diagram (say, λ) corresponds to one of the particles λ_j jumping to the right by one. Denote by $\mathcal{F}(\bar{\nu}, \lambda) \subset \{1, \dots, \ell + 1\}$ the set of all indices of upper particles λ_j which are *free to jump* (i.e., which can jump to the right without breaking the interlacing with lower particles). For example, on Fig. 5 we have $\mathcal{F}(\bar{\nu}, \lambda) = \{1, 2, 4\}$. Denote for $m \in \{1, 2, \dots\} \cup \{+\infty\}$:

$$\text{next}(m) := \max\{j \in \mathcal{F}(\bar{\nu}, \lambda) : j \leq m\}. \quad (5.14)$$

In words, this is the index of the first free particle to the right of λ_m (including λ_m).

Now we can complete the definition of the functions $W_{(\alpha)}^h$ and $V_{(\alpha)}^h$. The only nonzero value of $W_{(\alpha)}^h(\lambda, \cdot \mid \bar{\nu})$ is

$$W_{(\alpha)}^h(\lambda, \lambda + \mathbf{e}_{\text{next}(h)} \mid \bar{\nu}) = \alpha. \quad (5.15)$$

The values of $V_{(\alpha)}^h$ are given for any $m, j + 1 \in \mathcal{F}(\bar{\nu}, \lambda)$ by

$$V_{(\alpha)}^h(\lambda, \lambda + \mathbf{e}_m \mid \bar{\nu} - \bar{\mathbf{e}}_j, \bar{\nu}) = \mathbf{r}_j^h(\bar{\nu}, \lambda) \mathbf{1}_{m=\text{next}(j)} + (1 - \mathbf{r}_j^h(\bar{\nu}, \lambda)) \mathbf{1}_{m=j+1}, \quad (5.16)$$

with

$$\mathbf{r}_j^h(\bar{\nu}, \lambda) := \frac{1}{T_j(\bar{\nu}, \lambda \mid q, t)} \left(\sum_{i=1}^j S_i(\bar{\nu}, \lambda \mid q, t) - \sum_{i=1}^{j-1} T_i(\bar{\nu}, \lambda \mid q, t) - \mathbf{1}_{j \geq h} \right), \quad (5.17)$$

where the quantities S_i, T_i are defined by (5.9)–(5.10) (with the understanding that if $\lambda_i = \bar{\nu}_{i-1}$, i.e., the particle λ_i is blocked and cannot move to the right, then $S_i = T_{i-1} = 0$).

Proposition 5.3. *For each $h \in \{1, 2, \dots\} \cup \{+\infty\}$, the ‘rates of independent jumps’ $W_{(\alpha)}^h$ (5.15) and the ‘probabilities of triggered moves’ $V_{(\alpha)}^h$ (5.13), (5.16)–(5.17) define an RSK-type bivariate ‘dynamics’ $Q_{\mathbf{A};(\alpha)}^{(2)}$ via the construction explained in §4.4.*

Proof. This is proven in [16, §6.4.2 and §6.5.3]. \square

The ‘dynamics’ corresponding to $(W_{(\alpha)}^h, V_{(\alpha)}^h)$ defined by (5.15)–(5.17) above can be intuitively interpreted in the following way.¹⁸ Let $\begin{bmatrix} \lambda \\ \bar{\lambda} \end{bmatrix} \in \mathbb{Y}^{(2)}(\mathbf{A};(\alpha))$ denote the current state of the ‘dynamics’. We argue in terms of interlacing arrays, cf. Fig. 5.

The only particle that can jump on the upper level is $\lambda_{\text{next}(h)}$, and it jumps to the right by one according to an exponential clock of rate α .¹⁹ One can equivalently say that the particle λ_h itself tries to jump (with rate α), but if it is blocked, then it *donates* its jump to the first free particle to the right of itself.

On the lower level, if a particle $\bar{\lambda}_j$ moves to the right by one,²⁰ then it instantaneously pushes (to the right by one) its first free upper right neighbor $\lambda_{\text{next}(j)}$ with probability $r_j^h(\bar{\lambda} + \bar{\mathbf{e}}_j, \lambda)$ (5.17), or pulls (also to the right by one) its upper left neighbor λ_{j+1} with the complementary probability $1 - r_j^h(\bar{\lambda} + \bar{\mathbf{e}}_j, \lambda)$. See Fig. 6.

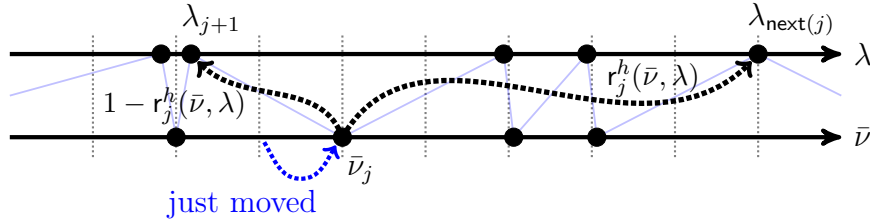


FIGURE 6. Pushing and pulling ‘probabilities’ in (5.16), $\bar{\nu}_j = \bar{\lambda}_j + 1$.

Remark 5.4. The ‘dynamics’ $Q_{\mathbf{A};(\alpha)}^{(2)}$ on interlacing configurations belongs to the class of so-called *nearest neighbor* ‘dynamics’, in which a moving lower level particle can (with some probabilities) either push its first free upper right neighbor, or pull its immediate upper left neighbor. The main result of [16] is a complete classification of nearest neighbor ‘dynamics’ of the form $Q_{\mathbf{A};(\alpha)}^{(2)}$, where \mathbf{A} is a finite length specialization (see (2.2)). It is possible to extend that classification to arbitrary (q, t) -nonnegative specializations \mathbf{A} , and also to the cases when the usual specialization (α) is replaced by a dual or a Plancherel one (see §5.3 and §5.5 below). We do not pursue this direction here.

¹⁸We are using probabilistic terms despite the fact that some ‘probabilities’ can be negative (see the beginning of §4 for more detail). When we speak about conditioning on an event which possibly can have negative probability, this should be understood as an intuitive appeal to the product rule (4.15) defining the jump rates via the quantities (5.15)–(5.17).

¹⁹Setting $h = +\infty$ means that the last (i.e., the leftmost) particle jumps independently.

²⁰We assume that the lower level particles evolve according to the univariate dynamics $Q_{\mathbf{A}}$. The functions $(W_{(\alpha)}^h, V_{(\alpha)}^h)$ then provide the necessary “induction step” leading to the upper univariate dynamics $Q_{\mathbf{A};(\alpha)}^{(2)}$.

5.3. Transposition, and bivariate ‘dynamics’ with a dual variable. Using a ‘dynamics’ with a usual variable α (§5.2) together with identities from §5.1, it is possible to construct a ‘dynamics’ $\mathbf{Q}_{\mathbf{A};(\beta)}^{(2)}$ with a dual variable $\beta > 0$ to the specialization.

Lemma 5.5. *The state space (4.6) of a bivariate ‘dynamics’ $\mathbf{Q}_{\mathbf{A};(\beta)}^{(2)}$ can be characterized as*

$$\mathbb{Y}^{(2)}(\mathbf{A}; (\beta)) = \left\{ \left[\begin{smallmatrix} \lambda \\ \bar{\lambda} \end{smallmatrix} \right] : \bar{\lambda} \in \mathbb{Y}(\mathbf{A}) \text{ and } \bar{\lambda} \prec_{\vee} \lambda \right\}. \quad (5.18)$$

Proof. Similar to the proof of Lemma 5.2. \square

To describe functions $(W_{(\beta)}, V_{(\beta)})$ satisfying (4.16), (4.17), and (4.18), let us rewrite (4.18) in terms of transposed Young diagrams:

Lemma 5.6. *For $\mathbf{B} = (\beta)$, identity (4.18) can be rewritten in the following equivalent form:*

$$\begin{aligned} \sum_{\bar{\square} \in \mathcal{D}(\bar{\nu})} V(\lambda, \lambda + \square \mid \bar{\nu} - \bar{\square}, \bar{\nu}) P_{\lambda' / (\bar{\nu} - \bar{\square})'}(\beta \mid t, q) \psi'_{\bar{\nu}' / (\bar{\nu} - \bar{\square})'}(t, q) \\ + \frac{1-t}{1-q} W(\lambda, \lambda + \square \mid \bar{\nu}) P_{\lambda' / \bar{\nu}'}(\beta \mid t, q) = P_{(\lambda + \square)' / \bar{\nu}'}(\beta \mid t, q) \psi'_{(\lambda + \square)' / \lambda'}(t, q), \end{aligned} \quad (5.19)$$

for all $\bar{\nu} \in \mathbb{Y}(\mathbf{A})$, $\lambda \in \mathbb{Y}(\mathbf{A} \cup (\beta))$, and all $\square \in \mathcal{U}(\lambda)$, such that $\left[\begin{smallmatrix} \lambda + \square \\ \bar{\nu} \end{smallmatrix} \right] \in \mathbb{Y}^{(2)}(\mathbf{A}; (\beta))$.

Note that (5.19) involves specializations of Macdonald polynomials into one *usual* variable equal to β .

Proof. Using identities from §5.1, we can rewrite expressions entering (4.18) as follows:

$$\begin{aligned} P_{\lambda / \bar{\nu} - \bar{\square}}(\hat{\beta} \mid q, t) \psi'_{\bar{\nu}' / \bar{\nu} - \bar{\square}}(q, t) &= Q_{\lambda' / (\bar{\nu} - \bar{\square})'}(\beta \mid t, q) \psi'_{\bar{\nu}' / (\bar{\nu} - \bar{\square})'}(t, q) \\ &= P_{\lambda' / (\bar{\nu} - \bar{\square})'}(\beta \mid t, q) \frac{b_{\lambda'}(t, q)}{b_{(\bar{\nu} - \bar{\square})'}(t, q)} \varphi_{\bar{\nu}' / (\bar{\nu} - \bar{\square})'}(t, q) \frac{b_{(\bar{\nu} - \bar{\square})'}(t, q)}{b_{\bar{\nu}'}(t, q)} \\ &= \frac{b_{\lambda'}(t, q)}{b_{\bar{\nu}'}(t, q)} P_{\lambda' / (\bar{\nu} - \bar{\square})'}(\beta \mid t, q) \frac{1-q}{1-t} \psi'_{\bar{\nu}' / (\bar{\nu} - \bar{\square})'}(t, q). \end{aligned}$$

In the last equality we have used (5.4) with parameters q and t interchanged, hence the coefficient $\frac{1-q}{1-t}$. Rewriting the other two coefficients in (4.18) in a similar way, one gets (5.19). \square

Proposition 5.7. *Functions $(W_{(\beta)}, V_{(\beta)})$ correspond (as in §4.4) to an RSK-type bivariate ‘dynamics’ $\mathbf{Q}_{\mathbf{A};(\beta)}^{(2)}$ with a dual variable β if and only if they have the form*

$$\begin{aligned} W_{(\beta)}(\lambda, \lambda + \square \mid \bar{\nu}) &= W_{(\alpha)}(\lambda', (\lambda + \square)' \mid \bar{\nu}') \Big|_{q \leftrightarrow t}, & \left[\begin{smallmatrix} \lambda \\ \bar{\nu} \end{smallmatrix} \right], \left[\begin{smallmatrix} \lambda + \square \\ \bar{\nu} \end{smallmatrix} \right] &\in \mathbb{Y}^{(2)}(\mathbf{A}; (\beta)); \\ V_{(\beta)}(\lambda, \lambda + \square \mid \bar{\nu} - \bar{\square}, \bar{\nu}) &= V_{(\alpha)}(\lambda', (\lambda + \square)' \mid (\bar{\nu} - \bar{\square})', \bar{\nu}') \Big|_{q \leftrightarrow t}, & \left[\begin{smallmatrix} \lambda \\ \bar{\nu} - \bar{\square} \end{smallmatrix} \right], \left[\begin{smallmatrix} \lambda + \square \\ \bar{\nu} \end{smallmatrix} \right] &\in \mathbb{Y}^{(2)}(\mathbf{A}; (\beta)); \end{aligned}$$

where $\bar{\square} \in \mathcal{D}(\bar{\nu})$, $\square \in \mathcal{U}(\lambda)$, and the functions $(W_{(\alpha)}, V_{(\alpha)})$ correspond to a bivariate ‘dynamics’ with the usual variable $\alpha := \frac{1-q}{1-t}\beta$, but in the setting with the swapped Macdonald parameters $(q, t) \rightarrow (t, q)$.

Proof. Readily follows from Lemma 5.6 and Theorem 4.6. \square

In words, Proposition 5.7 means that to construct a bivariate ‘dynamics’ with a dual variable β and parameters (q, t) , one can take a bivariate ‘dynamics’ with a *usual* variable $\frac{1-q}{1-t}\beta$ and *swapped* parameters (t, q) , and run the latter ‘dynamics’ in terms of *columns* of Young diagrams instead of rows.

Denote by $(W_{(\beta)}^h, V_{(\beta)}^h)$, where $h \in \{1, 2, \dots\} \cup \{+\infty\}$, the functions corresponding (via Proposition 5.7) to the functions $(W_{(\alpha)}^h, V_{(\alpha)}^h)$ from §5.2 (here $\alpha = \frac{1-q}{1-t}\beta$).

5.4. Plancherel specialization and Young graph. We will need two lemmas describing Plancherel specializations of skew Macdonald symmetric functions:

Lemma 5.8. *For any $\lambda, \kappa \in \mathbb{Y}$ with $\lambda \subseteq \kappa$, the expression $(|\kappa| - |\lambda|)! \cdot P_{\kappa/\lambda}(\mathbf{P}_1)$ is equal to the coefficient of Q_κ in the expansion of $p_1^{|\kappa|-|\lambda|} Q_\lambda$ with respect to the linear basis $\{Q_\mu\}_{\mu \in \mathbb{Y}}$ of the algebra Sym .*

Proof. Fix $\lambda \in \mathbb{Y}$, and put $\nu = \emptyset$ and $\mathbf{A} = \mathbf{P}_1$ in the skew Cauchy identity (4.12):

$$\sum_{\kappa \in \mathbb{Y}} P_{\kappa/\lambda}(\mathbf{P}_1) Q_\kappa(\mathbf{B}) = e^{p_1(\mathbf{B})} Q_\lambda(\mathbf{B})$$

(we also used (4.3)). Considering terms in both sides which are homogeneous with respect to \mathbf{B} , we get the following identities:

$$p_1^n Q_\lambda = \sum_{\kappa \in \mathbb{Y}_{|\lambda|+n}} n! P_{\kappa/\lambda}(\mathbf{P}_1) Q_\kappa, \quad n = 1, 2, \dots \quad (5.20)$$

(since \mathbf{B} is any specialization, we could write identities in the algebra Sym). \square

Lemma 5.9. *For $\lambda, \kappa \in \mathbb{Y}$ with $\lambda \subseteq \kappa$ and $n := |\kappa| - |\lambda|$, one has*

$$n! P_{\kappa/\lambda}(\mathbf{P}_1) = \sum_{\lambda = \mu^{(0)} \nearrow \mu^{(1)} \nearrow \dots \nearrow \mu^{(n-1)} \nearrow \mu^{(n)} = \kappa} \varphi'_{\mu^{(1)}/\mu^{(0)}} \varphi'_{\mu^{(2)}/\mu^{(1)}} \cdots \varphi'_{\mu^{(n)}/\mu^{(n-1)}}, \quad (5.21)$$

where the quantities $\varphi'_{\mu^{(i+1)}/\mu^{(i)}} = \varphi'_{\mu^{(i+1)}/\mu^{(i)}}(q, t)$ are defined in (5.1).

Proof. For $n = 1$, we have by Lemma 5.8:

$$p_1 Q_\lambda = \sum_{\square \in \mathcal{U}(\lambda)} P_{\lambda+\square/\lambda}(\mathbf{P}_1) Q_{\lambda+\square}.$$

Comparing this with the Pieri formula [50, VI.(6.24.iii)], we see that $P_{\lambda+\square/\lambda}(\mathbf{P}_1) = \varphi'_{\lambda+\square/\lambda}$. The general n statement is readily established by induction. \square

Remark 5.10. In particular, $P_{\lambda+\square/\lambda}(\mathbf{P}_1)$ is equal to the coefficient by the first power of β in $P_{\lambda+\square/\lambda}(\hat{\beta} \mid q, t)$, and also (by (5.4)) to $\frac{1-q}{1-t}$ times the coefficient by the first power of α in $P_{\lambda+\square/\lambda}(\alpha \mid q, t)$, see (5.1).

Lemmas 5.8 and 5.9 reflect the structure of the Young graph (= the lattice of all Young diagrams ordered by inclusion) with formal Macdonald (q, t) edge multiplicities: The multiplicity of the edge $\lambda \nearrow \lambda + \square$ is given by $P_{\lambda+\square/\lambda}(\mathbf{P}_1 \mid q, t) = \varphi'_{\lambda+\square/\lambda}(q, t)$. Moreover, for any $\lambda, \mu \in \mathbb{Y}$, the quantity $(|\lambda| - |\mu|)! \cdot P_{\lambda/\mu}(\mathbf{P}_1 \mid q, t)$ is equal to the total number of paths from μ to λ (counted with these edge multiplicities). See [45] and, e.g., [59, §9.1] for more detail.

The problem of classifying (q, t) -nonnegative specializations of the algebra \mathbf{Sym} (§2.3) is equivalent to classifying certain *coherent measures* on the Young graph with these edge multiplicities. The coherency property of a family of probability measures \mathcal{M}_n on \mathbb{Y}_n is formulated as

$$\sum_{\lambda: \lambda \searrow \mu} \mathcal{M}_n(\lambda) \frac{P_\mu(\mathbf{PI}_1 | q, t)}{n \cdot P_\lambda(\mathbf{PI}_1 | q, t)} \varphi'_{\lambda/\mu}(q, t) = \mathcal{M}_{n-1}(\mu) \quad \text{for all } n \geq 1 \text{ and all } \mu \in \mathbb{Y}_{n-1}.$$

One can readily check that the coherent measures defined in §3.1 satisfy this relation by reducing it to the Pieri formula [50, VI.(6.24.iii)] (hence the name for the measures (3.1)). See, e.g., [46], [12], [45] for more detail about coherent measures and boundaries of branching graphs.

Connection between the above coherency relation and stochastic links (4.4)–(4.5) is explained in [15] in the Schur ($q = t$) case, when it highlights the interplay between representation theory of the infinite symmetric group $S(\infty)$ and the infinite-dimensional unitary group $U(\infty)$.

In the Hall–Littlewood ($q = 0$) case, the above coherency property may be interpreted as coming from central measures on infinite uni-uppertriangular matrices over a finite field, see §1.1 and Remark 1.3 in particular.

5.5. Bivariate ‘dynamics’ with a Plancherel parameter. We will now construct bivariate ‘dynamics’ $\mathbf{Q}_{\mathbf{A}; \mathbf{PI}_\gamma}^{(2)}$, where \mathbf{A} is an arbitrary (q, t) -nonnegative specialization and $\gamma > 0$. First, observe that condition $\left[\frac{\lambda}{\bar{\lambda}}\right] \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{PI}_\gamma)$ means precisely that $\bar{\lambda} \in \mathbb{Y}(\mathbf{A})$ and $\bar{\lambda} \subseteq \lambda$ (this follows from Lemma 5.9, see also Lemmas 5.2 and 5.5). Our construction will involve two steps, “infinitesimal” and “general”.

5.5.1. Infinitesimal step. Let (W_\circ, V_\circ) be functions satisfying the following identities:

$$\sum_{\nu: \bar{\nu} \nearrow \nu} W_\circ(\bar{\nu}, \nu) = 1; \tag{5.22}$$

$$\sum_{\nu: \lambda \nearrow \nu, \bar{\nu} \nearrow \nu} V_\circ(\lambda, \nu | \bar{\lambda}, \bar{\nu}) = 1, \quad \bar{\lambda} \nearrow \lambda, \quad \bar{\lambda} \nearrow \bar{\nu}; \tag{5.23}$$

$$\sum_{\bar{\lambda}: \bar{\lambda} \nearrow \bar{\nu}} V_\circ(\lambda, \nu | \bar{\lambda}, \bar{\nu}) \psi_{\lambda/\bar{\lambda}} \psi'_{\bar{\nu}/\bar{\lambda}} + W_\circ(\lambda, \nu) \mathbf{1}_{\lambda=\bar{\nu}} = \psi_{\nu/\bar{\nu}} \psi'_{\nu/\lambda}, \quad \bar{\nu} \nearrow \nu, \quad \lambda \nearrow \nu. \tag{5.24}$$

Note that (5.24) can be viewed as an infinitesimal version of the general identity (4.18) corresponding to taking $\mathbf{B} = (\alpha)$ or (β) and considering the coefficient by the first power of α or β , respectively (cf. Remark 5.10).

One can choose (W_\circ, V_\circ) using the functions from §5.2 and §5.3:

$$V_\circ^{\alpha; h}(\lambda, \nu | \bar{\lambda}, \bar{\nu}) := V_{(\alpha)}^h(\lambda, \nu | \bar{\lambda}, \bar{\nu}), \quad W_\circ^{\alpha; h}(\lambda, \nu) := \frac{1}{p_1(\alpha)} W_{(\alpha)}^h(\lambda, \nu | \lambda), \tag{5.25}$$

or

$$V_\circ^{\beta; h}(\lambda, \nu | \bar{\lambda}, \bar{\nu}) := V_{(\beta)}^h(\lambda, \nu | \bar{\lambda}, \bar{\nu}), \quad W_\circ^{\beta; h}(\lambda, \nu) := \frac{1}{p_1(\hat{\beta})} W_{(\beta)}^h(\lambda, \nu | \lambda). \tag{5.26}$$

(Each of the two families (W_\circ^h, V_\circ^h) depends on $h \in \{1, 2, \dots\} \cup \{+\infty\}$.) Then (4.18) readily implies identity (5.24). Observe that in both cases the values of the functions (W_\circ, V_\circ) do not depend on the parameters α, β .

Remark 5.11. One should think that the choice (5.25) corresponds to the “row insertion”, while (5.26) leads to the “column insertion”, cf. the idea of transposing Young diagrams employed in §5.3 above (see also connections to the classical RSK insertion algorithms discussed in [16, §7]). Different choices (5.25) and (5.26) lead to different ‘dynamics’ with a Plancherel parameter.

5.5.2. *General step: construction of the ‘dynamics’.* Assume now that we have chosen functions (W_\circ, V_\circ) as in §5.5.1. We will now explain the construction of functions $W_{\mathbf{P}_\gamma}(\lambda, \cdot | \bar{\lambda})$ and $V_{\mathbf{P}_\gamma}(\lambda, \cdot | \bar{\lambda}, \cdot)$ corresponding to the desired bivariate ‘dynamics’ $\mathbf{Q}_{\mathbf{A}; \mathbf{P}_\gamma}^{(2)}$. Assume that the pair $\left[\frac{\lambda}{\bar{\lambda}}\right] \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{P}_\gamma)$ is fixed, and denote $n := |\lambda| - |\bar{\lambda}|$. Our construction of the desired functions is probabilistic and consists of the following steps:

- (1) Sample random intermediate Young diagrams

$$\bar{\lambda} = \mu^{(0)} \nearrow \mu^{(1)} \nearrow \mu^{(2)} \nearrow \dots \nearrow \mu^{(n)} = \lambda$$

according to the distribution (cf. Lemma 5.9)

$$\text{Prob}(\mu^{(i)} : i = 0, 1, \dots, n) = \frac{\varphi'_{\mu^{(1)}/\mu^{(0)}} \varphi'_{\mu^{(2)}/\mu^{(1)}} \cdots \varphi'_{\mu^{(n)}/\mu^{(n-1)}}}{n! P_{\lambda/\bar{\lambda}}(\mathbf{P}_1)}. \quad (5.27)$$

Let us add auxiliary Young diagrams $\mu^{(i+\frac{1}{2})}$, $i = 0, 1, \dots, n$ as follows:

$$\bar{\lambda} = \mu^{(0)} = \mu^{(\frac{1}{2})} \nearrow \mu^{(1)} = \mu^{(\frac{3}{2})} \nearrow \mu^{(2)} = \mu^{(\frac{5}{2})} \nearrow \dots \nearrow \mu^{(n)} = \mu^{(n+\frac{1}{2})} = \lambda.$$

- (2) Independent jumps $\lambda \rightarrow \lambda + \square$ happen according to an exponential clock with rate $p_1(\mathbf{P}_\gamma) = \gamma^{\frac{1-q}{1-t}}$. When this clock rings, we pick a uniformly random number $m \in \{\frac{1}{2}, \frac{3}{2}, \dots, n + \frac{1}{2}\}$, and add a box $\square^{(m)}$ to $\mu^{(m)}$ with ‘probability’

$$\text{‘Prob’}(\mu^{(m)} \rightarrow \mu^{(m)} + \square^{(m)}) = W_\circ(\mu^{(m)}, \mu^{(m)} + \square^{(m)}).$$

If $m \leq n - \frac{1}{2}$, any such move will propagate to all the higher levels according to the next rule.

- (3) Any move happening at any level (recall that the bottommost diagram $\bar{\lambda} = \mu^{(0)}$ itself evolves according to the univariate dynamics $\mathbf{Q}_\mathbf{A}$)

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots, n - \frac{1}{2}, n, n + \frac{1}{2}$$

almost surely propagates all the way to the uppermost diagram $\lambda = \mu^{(n+\frac{1}{2})}$ according to the conditional ‘probabilities’

$$\begin{aligned} \text{‘Prob’}(\mu^{(i+\frac{1}{2})} \rightarrow \mu^{(i+\frac{1}{2})} + \square^{(i+\frac{1}{2})} | \mu^{(i)} \rightarrow \mu^{(i)} + \square^{(i)}) \\ = V_\circ(\mu^{(i+\frac{1}{2})}, \mu^{(i+\frac{1}{2})} + \square^{(i+\frac{1}{2})} | \mu^{(i)}, \mu^{(i)} + \square^{(i)}), \quad i = j, j + \frac{1}{2}, j + 1, \dots, n - \frac{1}{2}, n. \end{aligned}$$

Note that if $\mu^{(i+\frac{1}{2})} = \mu^{(i)}$ above, then it must be $\square^{(i+\frac{1}{2})} = \square^{(i)}$ (i.e., the above probability is equal to $\mathbf{1}_{\square^{(i+\frac{1}{2})} = \square^{(i)}}$). This is similar to the short-range pushing mechanism, cf. (5.13).

Averaging over the $\mu^{(\cdot)}$ ’s with distribution (5.27), one arrives at certain functions $W_{\mathbf{P}_\gamma}(\lambda, \cdot | \bar{\lambda})$ and $V_{\mathbf{P}_\gamma}(\lambda, \cdot | \bar{\lambda}, \cdot)$ describing independent jumps and triggered moves for the pair $\left[\frac{\lambda}{\bar{\lambda}}\right]$.

Remark 5.12. It is worth noting that the above construction describes the evolution during a small time interval. In particular, each jump of the bivariate ‘dynamics’ requires sampling the intermediate Young diagrams $\mu^{(i)}$ again. In fact, it is possible to formulate the ‘dynamics’

without such an excessive sampling of intermediate diagrams (with the help of continuous levels $\mu^{(s)}$, where $s \in [0, 1]$). We do this (in a slightly different language) in §6.5 below.

Proposition 5.13. *Thus defined functions $(W_{\mathbf{P}_{\mathbf{I}_\gamma}}, V_{\mathbf{P}_{\mathbf{I}_\gamma}})$ correspond (as in §4.4) to an RSK-type bivariate ‘dynamics’ $\mathbf{Q}_{\mathbf{A}; \mathbf{P}_{\mathbf{I}_\gamma}}^{(2)}$ with the Plancherel parameter $\gamma > 0$.*

Proof. We will check that these functions satisfy (4.16), (4.17), and (4.18) with $\mathbf{B} = \mathbf{P}_{\mathbf{I}_\gamma}$. The first and second identities are straightforward. So, we must show that (see (4.18))

$$\begin{aligned} \sum_{\bar{\square} \in \mathcal{D}(\bar{\nu})} V_{\mathbf{P}_{\mathbf{I}_\gamma}}(\lambda, \lambda + \square \mid \bar{\nu} - \bar{\square}, \bar{\nu}) P_{\lambda/\bar{\nu} - \bar{\square}}(\mathbf{P}_{\mathbf{I}_1}) \psi'_{\bar{\nu} - \bar{\square}} \\ + \gamma^{-1} W_{\mathbf{P}_{\mathbf{I}_\gamma}}(\lambda, \lambda + \square \mid \bar{\nu}) P_{\lambda/\bar{\nu}}(\mathbf{P}_{\mathbf{I}_1}) = P_{\lambda + \square/\bar{\nu}}(\mathbf{P}_{\mathbf{I}_1}) \psi'_{\lambda + \square/\bar{\nu}}, \end{aligned} \quad (5.28)$$

for all $\bar{\nu} \in \mathbb{Y}(\mathbf{A})$, $\lambda \in \mathbb{Y}(\mathbf{A} \cup \mathbf{B})$, and all $\square \in \mathcal{U}(\lambda)$, such that $\begin{bmatrix} \lambda + \square \\ \bar{\nu} \end{bmatrix} \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$.

To simplify the argument, let us establish (5.28) in the simplest nontrivial case $|\lambda| - |\bar{\nu}| = 1$. The general case is analogous.

We can write

$$V_{\mathbf{P}_{\mathbf{I}_\gamma}}(\lambda, \nu \mid \bar{\lambda}, \bar{\nu}) = \sum_{\mu^{(1)}} \frac{\varphi'_{\mu^{(1)}/\bar{\lambda}} \varphi'_{\lambda/\mu^{(1)}}}{2P_{\lambda/\bar{\lambda}}(\mathbf{P}_{\mathbf{I}_1})} \sum_{\kappa: \kappa \searrow \mu^{(1)}} V_{\circ}(\lambda, \nu \mid \mu^{(1)}, \kappa) V_{\circ}(\mu^{(1)}, \kappa \mid \bar{\lambda}, \bar{\nu}), \quad \bar{\lambda} \nearrow \bar{\nu}, \lambda \nearrow \nu. \quad (5.29)$$

Indeed, in the definition of $V_{\mathbf{P}_{\mathbf{I}_\gamma}}$ above we have $n = |\lambda| - |\bar{\lambda}| = 2$. In the summation, κ is the new state of the diagram $\mu^{(1)}$, which means that the whole transition looks as $\begin{bmatrix} \lambda \\ \mu^{(1)} \\ \bar{\lambda} \end{bmatrix} \rightarrow \begin{bmatrix} \nu \\ \kappa \\ \bar{\nu} \end{bmatrix}$.

On the other hand, the independent jump ‘rate’ $W_{\mathbf{P}_{\mathbf{I}_\gamma}}$ is given by

$$W_{\mathbf{P}_{\mathbf{I}_\gamma}}(\lambda, \nu \mid \bar{\nu}) = \gamma \frac{1-q}{1-t} \left(\frac{1}{2} \sum_{\kappa} W_{\circ}(\bar{\nu}, \kappa) V_{\circ}(\lambda, \nu \mid \bar{\nu}, \kappa) + \frac{1}{2} W_{\circ}(\lambda, \nu \mid \bar{\nu}) \right). \quad (5.30)$$

Indeed, this time $n = |\lambda| - |\bar{\nu}| = 1$ in the definition of $W_{\mathbf{P}_{\mathbf{I}_\gamma}}$. The diagram κ now represents the new state of $\mu^{(\frac{1}{2})}$. The sum over κ corresponds to an independent jump of $\mu^{(\frac{1}{2})}$, and the second summand corresponds to an independent jump of $\mu^{(\frac{3}{2})}$. The factor $\gamma \frac{1-q}{1-t}$ is simply the total rate of an independent jump of λ in the pair $\begin{bmatrix} \lambda \\ \bar{\nu} \end{bmatrix}$.

Plugging (5.29) into a part (5.28), we obtain

$$\begin{aligned} \sum_{\bar{\square} \in \mathcal{D}(\bar{\nu})} V_{\mathbf{P}_{\mathbf{I}_\gamma}}(\lambda, \lambda + \square \mid \bar{\nu} - \bar{\square}, \bar{\nu}) P_{\lambda/\bar{\nu} - \bar{\square}}(\mathbf{P}_{\mathbf{I}_1}) \psi'_{\bar{\nu} - \bar{\square}} \\ = \sum_{\bar{\square}, \mu^{(1)}, \kappa} \frac{1}{2} \varphi'_{\mu^{(1)}/\bar{\nu} - \bar{\square}} \varphi'_{\lambda/\mu^{(1)}} V_{\circ}(\lambda, \lambda + \square \mid \mu^{(1)}, \kappa) V_{\circ}(\mu^{(1)}, \kappa \mid \bar{\nu} - \bar{\square}, \bar{\nu}) \psi'_{\bar{\nu} - \bar{\square}} \\ = \sum_{\mu^{(1)}, \kappa} \frac{1}{2} \varphi'_{\lambda/\mu^{(1)}} V_{\circ}(\lambda, \lambda + \square \mid \mu^{(1)}, \kappa) \left[\varphi'_{\kappa/\bar{\nu}} \psi'_{\kappa/\mu^{(1)}} - \frac{1-q}{1-t} W_{\circ}(\mu^{(1)}, \kappa) \mathbf{1}_{\mu^{(1)} = \bar{\nu}} \right] \\ = \sum_{\mu^{(1)}, \kappa} \frac{1}{2} \varphi'_{\lambda/\mu^{(1)}} V_{\circ}(\lambda, \lambda + \square \mid \mu^{(1)}, \kappa) \varphi'_{\kappa/\bar{\nu}} \psi'_{\kappa/\mu^{(1)}} - \frac{1-q}{1-t} \sum_{\kappa} \frac{1}{2} \varphi'_{\lambda/\bar{\nu}} V_{\circ}(\lambda, \lambda + \square \mid \bar{\nu}, \kappa) W_{\circ}(\bar{\nu}, \kappa) \end{aligned}$$

$$\begin{aligned}
&= \sum_{\kappa} \frac{1}{2} \varphi'_{\kappa/\bar{\nu}} \varphi'_{\lambda+\square/\kappa} \psi'_{\lambda+\square/\lambda} \\
&\quad - \frac{1-q}{1-t} \left[\frac{1}{2} \varphi'_{\lambda/\bar{\nu}} W_{\circ}(\lambda, \lambda + \square) + \sum_{\kappa} \frac{1}{2} \varphi'_{\lambda/\bar{\nu}} V_{\circ}(\lambda, \lambda + \square \mid \bar{\nu}, \kappa) W_{\circ}(\bar{\nu}, \kappa) \right]
\end{aligned}$$

(using (5.24) and (5.4), we summed over \square , and then over $\mu^{(1)}$). By Lemma 5.9, we see that the first summand above is equal to $\psi'_{\lambda+\square/\lambda} P_{\lambda+\square/\bar{\nu}}(\mathbf{P}_1)$, which is the right-hand side of (5.28). The two remaining summands above cancel with

$$\gamma^{-1} W_{\mathbf{P}_1}(\lambda, \lambda + \square \mid \bar{\nu}) P_{\lambda/\bar{\nu}}(\mathbf{P}_1) = \gamma^{-1} W_{\mathbf{P}_1}(\lambda, \lambda + \square \mid \bar{\nu}) \varphi'_{\lambda/\bar{\nu}},$$

see (5.30). This concludes the proof. \square

Remark 5.14. One can readily see that the functions $(W_{\mathbf{B}}, V_{\mathbf{B}})$ constructed above (corresponding to bivariate ‘dynamics’ $Q_{\mathbf{A};\mathbf{B}}^{(2)}$, where \mathbf{B} is (α) , (β) , or \mathbf{P}_1) do not depend on the ‘lower’ specialization \mathbf{A} , cf. Remark 4.7.(3). This is the reason why we didn’t include \mathbf{A} in the notation.

6. RSK-TYPE ALGORITHM FOR SAMPLING HL-COHERENT MEASURES

From now on we will assume that the Macdonald parameter q is zero.²¹ For $q = 0$, we construct a randomized algorithm for sampling HL-coherent measures on Young diagrams (§3.1) which is an honest probabilistic object, that is, involves only nonnegative probabilities.

In contrast with the setting of §4 and §5, the discussion of sampling algorithms is simpler in the language of de-poissonized measures $\mathcal{HL}_n^{\alpha;\beta;\mathbf{P}_1}$ (cf. Remark 3.3). One can also readily describe the poissonized version of the algorithm, see Remark 6.17 below. It is worth noting that this does not affect the statement of the Law of Large Numbers (Theorem 1.4).

6.1. t -quantities. First, we need to understand how the quantities $T_i(\bar{\nu}, \lambda \mid q, t)$ and $S_j(\bar{\nu}, \lambda \mid q, t)$ (5.9)–(5.10) look like when we take parameters $(0, t)$. Recall that they are defined for $\bar{\nu} \prec_{\mathbf{h}} \lambda$. Denote $\ell := \ell(\bar{\nu})$, so (possibly appending λ by zeroes) we may think that $\ell(\lambda) = \ell + 1$.

Proposition 6.1. *For each fixed $i = 1, \dots, \ell$, the quantity $T_i(\bar{\nu}, \lambda \mid 0, t)$ is determined according to the following rules.*

(T0) *If $\bar{\nu}_i = \lambda_{i+1}$, then $T_i(\bar{\nu}, \lambda \mid 0, t) = 0$. If $\lambda_i > \bar{\nu}_i$ and $\lambda_{i+1} < \bar{\nu}_i - 1$, then $T_i(\bar{\nu}, \lambda \mid 0, t) = 1$. Otherwise, denote*

$$R := (\text{multiplicity of } \bar{\nu}_i \text{ in } \bar{\nu}) - 1, \quad L := (\text{multiplicity of } \bar{\nu}_i - 1 \text{ in } \bar{\nu})$$

(any of these numbers can be zero). One checks that the multiplicity of $\bar{\nu}_i$ in λ is either R or $R + 1$, and the multiplicity of $\bar{\nu}_i - 1$ in λ is either L or $L + 1$ (see Fig. 7). The value of T_i in each of the four cases is given by

(T1) *For (L, R) , we have $T_i(\bar{\nu}, \lambda \mid 0, t) = (1 - t^{L+1})/(1 - t)$.*

(T2) *For $(L, R + 1)$, we have $T_i(\bar{\nu}, \lambda \mid 0, t) = (1 - t^{L+1})(1 - t^{R+1})/(1 - t)$.*

(T3) *For $(L + 1, R)$, we have $T_i(\bar{\nu}, \lambda \mid 0, t) = 1/(1 - t)$.*

²¹It is possible to develop randomized ‘sampling’ algorithms (i.e., formal Markov ‘dynamics’ with negative probabilities of certain elements in a ‘transition matrix’) for the general parameters (q, t) by analogy, but we will not pursue this direction here.

(T4) For $(L + 1, R + 1)$, we have $T_i(\bar{\nu}, \lambda | 0, t) = (1 - t^{R+1})/(1 - t)$.

Cases (T0)–(T4) exhaust all possible configurations.

Proof. Direct $q = 0$ substitution in formula (5.9). \square

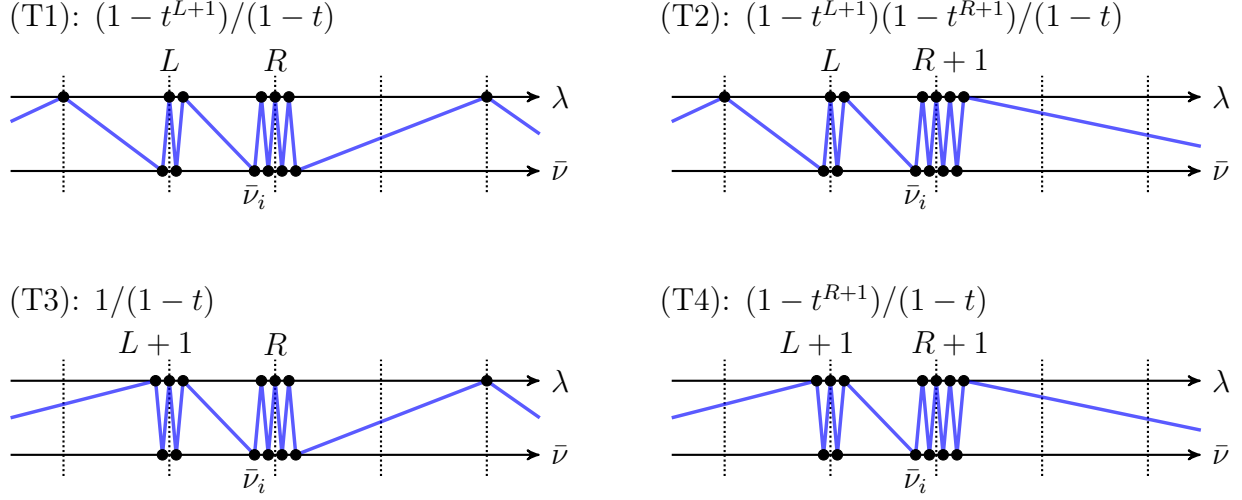


FIGURE 7. Cases (T1)–(T4) used to determine the value of $T_i(\bar{\nu}, \lambda | 0, t)$. On the picture we have $R = 3$ and $L = 2$. Young diagrams $\bar{\nu} \prec_h \lambda$ are represented by interlacing particle configurations.

Proposition 6.2. For each fixed $j = 1, \dots, \ell + 1$, the quantity $S_j(\bar{\nu}, \lambda | 0, t)$ is determined according to the following rules.

(S0) If $\lambda_j = \bar{\nu}_{j-1}$, then $S_j(\bar{\nu}, \lambda | 0, t) = 0$. If $\lambda_j > \bar{\nu}_j$ and $\lambda_j < \bar{\nu}_{j-1} - 1$, then $S_j(\bar{\nu}, \lambda | 0, t) = 1$.

Otherwise, denote

$$R := (\text{multiplicity of } \lambda_j + 1 \text{ in } \lambda), \quad L := (\text{multiplicity of } \lambda_j \text{ in } \lambda) - 1$$

(any of these numbers can be zero). Clearly, the multiplicity of $\lambda_j + 1$ in $\bar{\nu}$ can be either R or $R + 1$, and the multiplicity of λ_j in $\bar{\nu}$ is either L or $L + 1$ (see Fig. 8). The value of S_j in each of the four cases is given by

(S1) For (L, R) , we have $S_j(\bar{\nu}, \lambda | 0, t) = (1 - t^{R+1})/(1 - t)$.

(S2) For $(L, R + 1)$, we have $S_j(\bar{\nu}, \lambda | 0, t) = 1/(1 - t)$.

(S3) For $(L + 1, R)$, we have $S_j(\bar{\nu}, \lambda | 0, t) = (1 - t^{L+1})(1 - t^{R+1})/(1 - t)$.

(S4) For $(L + 1, R + 1)$, we have $S_j(\bar{\nu}, \lambda | 0, t) = (1 - t^{L+1})/(1 - t)$.

Cases (S0)–(S4) exhaust all possible configurations.

Proof. Direct $q = 0$ substitution in formula (5.10). \square

Remark 6.3. Observe that cases (T1), (T2), (T3), and (T4) on Fig. 7 describing the configuration around the particle $\bar{\nu}_i$ correspond to cases (S4), (S3), (S2), and (S1), respectively, for the configuration around the particle λ_{i+1} (see Fig. 8).

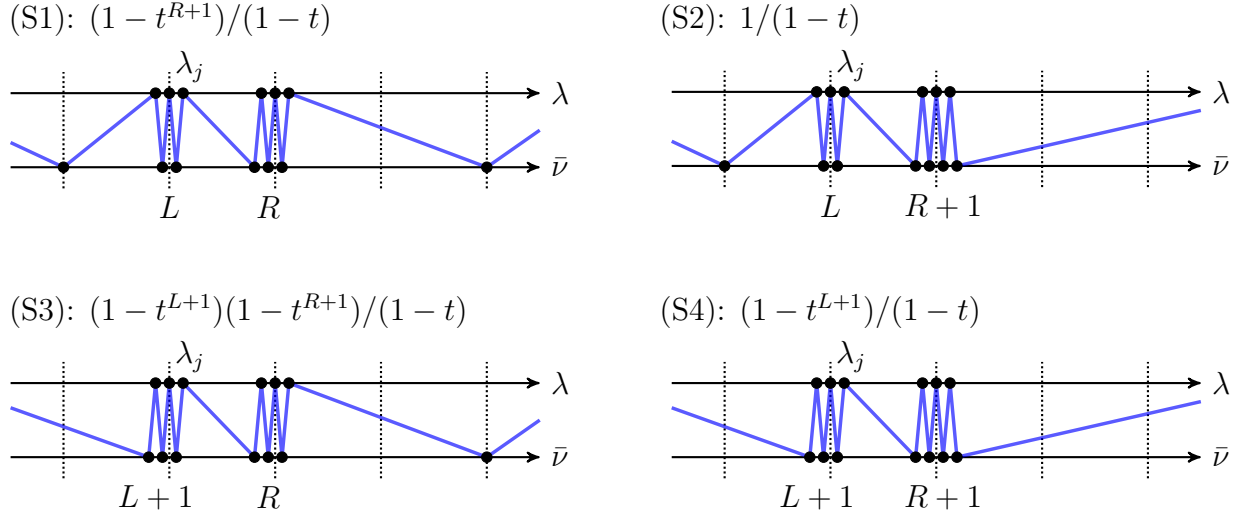


FIGURE 8. Cases (S1)–(S4) used to determine the value of $S_j(\bar{\nu}, \lambda | 0, t)$. On the picture we have $R = 3$ and $L = 2$.

For purposes of ‘dynamics’ with a dual variable (§5.3), we will also need the same quantities $T_i(\bar{\nu}, \lambda | q, t)$ and $S_j(\bar{\nu}, \lambda | q, t)$ (5.9)–(5.10) with parameters $(t, 0)$:

Proposition 6.4 ([16, §8.1]). *Let $\bar{\nu} \prec_h \lambda$, $\ell(\bar{\nu}) = \ell$, $\ell(\lambda) = \ell + 1$. Then*

$$T_i(\bar{\nu}, \lambda | t, 0) = \frac{(1 - t^{\bar{\nu}_i - \lambda_{i+1}})(1 - t^{\bar{\nu}_{i-1} - \bar{\nu}_i + 1} \mathbf{1}_{i > 1})}{1 - t^{\lambda_i - \bar{\nu}_i + 1}};$$

$$S_j(\bar{\nu}, \lambda | t, 0) = \frac{(1 - t^{\bar{\nu}_j - 1 - \lambda_j} \mathbf{1}_{j > 1})(1 - t^{\lambda_j - \lambda_{j+1} + 1} \mathbf{1}_{j < \ell + 1})}{1 - t^{\lambda_j - \bar{\nu}_j + 1} \mathbf{1}_{j < \ell + 1}},$$

where $i = 1, \dots, \ell$ and $j = 1, \dots, \ell + 1$.

6.2. Pushing ‘probabilities’. Let us write down explicit formulas for the pushing ‘probabilities’ $r_j^h(\bar{\nu}, \lambda)$ (5.17) in the cases of Macdonald parameters $(0, t)$ and $(t, 0)$. Here $h \in \{1, 2, \dots\} \cup \{+\infty\}$ is an additional parameter as before.

Recall (5.16) that the ‘probabilities’ $r_j^h(\bar{\nu}, \lambda)$ are defined for all $j \in \{1, \dots, \ell(\bar{\nu})\}$ such that $\bar{\nu}_j > \lambda_{j+1}$. For each such j , $r_j^h(\bar{\nu}, \lambda)$ represents the ‘probability’ that the particle $\bar{\nu}_j$ which has just moved on the lower level will push its first free upper right neighbor. With the complement ‘probability’ $1 - r_j^h(\bar{\nu}, \lambda)$, the particle $\bar{\nu}_j$ will pull its upper left neighbor λ_{j+1} . See Fig. 6 above.

From (5.17) one has that $r_j^h(\bar{\nu}, \lambda) = r_j^{+\infty}(\bar{\nu}, \lambda) - T_j^{-1}(\bar{\nu}, \lambda | q, t) \mathbf{1}_{j \geq h}$. For the Macdonald parameters $(0, t)$, the quantities $r_j^{+\infty}$ have the following form:

Proposition 6.5. *Assume that $\bar{\nu} \prec_h \lambda$, and $j = 1, \dots, \ell(\bar{\nu})$ is such that $\bar{\nu}_j > \lambda_{j+1}$. Denote*

$$D := (\text{multiplicity of } \bar{\nu}_j - 1 \text{ in } \bar{\nu}), \quad U := (\text{multiplicity of } \bar{\nu}_j - 1 \text{ in } \lambda)$$

(any of the numbers can be zero). Clearly, $U = D$ or $U = D + 1$ (see Fig. 9). There are two cases:

(r1) If $U = D$, then $r_j^{+\infty}(\bar{\nu}, \lambda | 0, t) = (1 - t)/(1 - t^{D+1})$.

(r2) If $U = D + 1$, then $r_j^{+\infty}(\bar{\nu}, \lambda | 0, t) = 1 - t$.

Cases (r1)–(r2) exhaust all possible configurations. Note that when particles are apart, more precisely, when $\lambda_{j+1} < \bar{\nu}_j - 1$, then $r_j^{+\infty}(\bar{\nu}, \lambda | 0, t) = 1$ by (r1).

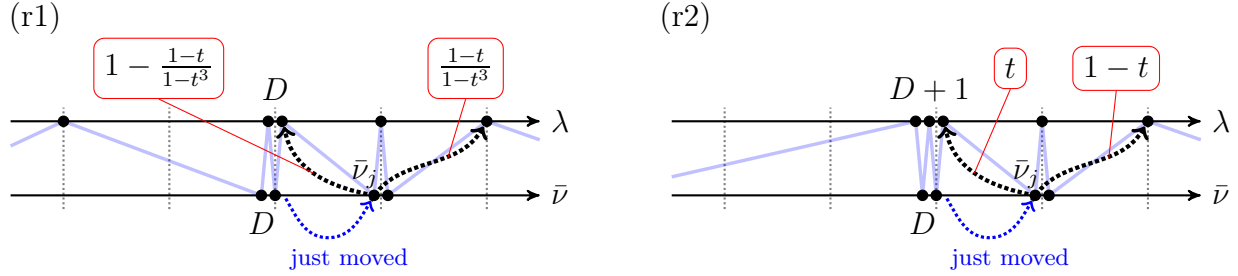
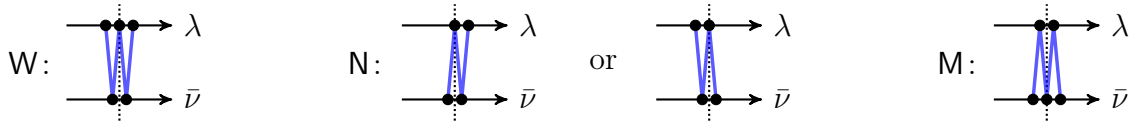


FIGURE 9. Cases (r1)–(r2) used to determine the value of $r_j^{+\infty}(\bar{\nu}, \lambda | 0, t)$. On the picture we have $D = 2$.

Proof. Let us represent the interlacing configuration $\bar{\nu} \prec_h \lambda$ as the union of blocks of particles sitting at the same position. There are three possible types of such blocks depending on the difference between the number of particles on the upper and the lower levels (notation reflects typical shape of zigzags; we understand that N could also mean N):



If there are k particles on the lower level, then we will denote such block by W_k , N_k , or M_k , respectively (for W_k , k is allowed to be zero). There are 7 ways in which two blocks can follow one another (the distance between the blocks is arbitrary):

$$WM \quad WN \quad NW \quad NM \quad NN \quad MW \quad MN \quad (6.1)$$

On the other hand, $r_j^{+\infty} = T_j^{-1}((S_1 - T_0) + (S_2 - T_1) + \dots + (S_j - T_{j-1}))$ (5.17) is determined by accumulating successive differences $S_i - T_{i-1}$, where i runs over all indices from 1 to j for which λ_i is free to move to the right (by agreement, $T_0 \equiv 0$). Nontrivial such differences arise when λ_i is the rightmost particle in one of the blocks W_b or N_b while $\bar{\nu}_{i-1}$ is the leftmost particle in M_a or N_a to the right. This reduces the number of combinations from 7 in (6.1) to 4. Using Propositions 6.1 and 6.2, one readily computes the corresponding differences $S_i - T_{i-1}$:

$$\begin{array}{c|c|c|c|c} & W_b M_a & W_b N_a & N_b M_a & N_b N_a \\ \hline S_i - T_{i-1} & 0 & t^a & -t^b & t^a - t^b \end{array} \quad (6.2)$$

(the difference $S_i - T_{i-1}$ does not depend on the distance between the blocks which can be arbitrary). For $i = 1$, there could be only two configurations, $W_b N_0$ or $N_b N_0$, for which $S_1 = 1$ and $1 - t^b$, respectively. This agrees with (6.2).

Let us now explain how one can compute the quantities $r_j^{+\infty}$. In the case (r1), the particle λ_{j+1} belongs to a block of type N which can be followed by either M or N on the right. The case (r2) consists of two other possibilities, WM and WN (λ_{j+1} belongs to W). See Fig. 9.

Let us consider the case $\mathbf{N}_b \mathbf{N}_a$, $\lambda_{j+1} \in \mathbf{N}_b$, $\bar{\nu}_j \in \mathbf{N}_a$. Using (6.2), one checks that (thanks to successive cancellations), $S_1 + (S_2 - T_1) + \dots + (S_j - T_{j-1})$ is equal to $1 - t^a$ regardless of the configuration of blocks to the right of \mathbf{N}_a . In fact, due to interlacing, any such configuration must have the same number of \mathbf{W} and \mathbf{M} blocks (we exclude the situation $\mathbf{N}_b \mathbf{N}_a$ because it does not contribute a nontrivial difference $S_i - T_{i-1}$). Moreover, by Proposition 6.1, we have $T_j = (1 - t^{b+1})(1 - t^a)/(1 - t)$, which yields $r_j^{+\infty} = (1 - t)/(1 - t^{b+1})$, as desired. The three remaining cases are analogous. \square

For the Macdonald parameters $(t, 0)$, it is convenient to represent $r_j^h(\bar{\nu}, \lambda) = r_j^1(\bar{\nu}, \lambda) + T_j^{-1}(\bar{\nu}, \lambda | q, t) \mathbf{1}_{j < h}$ (see (5.17)). The quantities r_j^1 are given in the next proposition:

Proposition 6.6 ([16, §8.2]). *Assume that $\bar{\nu} \prec_h \lambda$, and let $j = 1, \dots, \ell(\bar{\nu})$ be such that $\bar{\nu}_j > \lambda_{j+1}$. Then*

$$r_j^1(\bar{\nu}, \lambda | t, 0) = t^{\lambda_j - \bar{\nu}_j + 1} \frac{1 - t^{\bar{\nu}_{j-1} - \lambda_j} \mathbf{1}_{j > 1}}{1 - t^{\bar{\nu}_{j-1} - \bar{\nu}_j + 1} \mathbf{1}_{j > 1}}.$$

The cases when the quantities r_j^h are between 0 and 1 (i.e., when they are honest probabilities) can also be readily described:

Proposition 6.7. *For the Macdonald parameters $(0, t)$,*

$$0 \leq r_j^{+\infty}(\bar{\nu}, \lambda | 0, t) \leq 1 \quad \text{for all possible } \bar{\nu} \prec_h \lambda \text{ and } j,$$

and for other $h = 1, 2, \dots$, it can happen that $r_j^h(\bar{\nu}, \lambda | 0, t)$ is negative.

For the Macdonald parameters $(t, 0)$,

$$0 \leq r_j^1(\bar{\nu}, \lambda | t, 0) \leq 1 \quad \text{for all possible } \bar{\nu} \prec_h \lambda \text{ and } j,$$

and for other $h = 2, 3, \dots; +\infty$, it can happen that $r_j^h(\bar{\nu}, \lambda | t, 0)$ is greater than 1.

Proof. Straightforward verification. \square

6.3. Preliminary sampling algorithms. To better understand the desired sampling algorithm presented in §6.5 below, let us begin with two simpler constructions involving univariate and bivariate dynamics.

A “trivial” way to sample the measure $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{P}\mathbf{I}_\gamma}$ is to use univariate dynamics (§4.2) as follows. Let us write $\mathbf{A} = (\alpha; \beta; \mathbf{P}\mathbf{I}_\gamma)$ for short.

Sampling algorithm 1. *Start with an empty Young diagram $\lambda(0) = \emptyset$. At each step $k = 1, \dots, n$, add to the current Young diagram $\lambda = \lambda(k-1)$ one of the boxes $\square \in \mathcal{U}(\lambda)$ with probability*

$$\text{Prob}(\lambda \rightarrow \lambda + \square) = \frac{1}{p_1(\mathbf{A})} \frac{P_{\lambda + \square}(\mathbf{A} | 0, t)}{P_\lambda(\mathbf{A} | 0, t)} \psi'_{\lambda + \square / \lambda}(0, t). \quad (6.3)$$

Then the distribution of $\lambda(n)$ is $\mathcal{H}\mathcal{L}_n^{\mathbf{A}}$.

Indeed, under $\mathbf{Q}_{\mathbf{A}}$ boxes are added to the Young diagram (with probabilities (6.3)) in continuous time according to a Poisson process of rate $p_1(\mathbf{A})$. Conditioning on the event that there are exactly n points of this Poisson process during the time segment $[0, \tau]$, we arrive at Algorithm 1 producing the measure $\mathcal{H}\mathcal{L}_n^{\mathbf{A}}$. The fact that this algorithm works follows from Remark 4.2 and property (4.10).

The probabilities (6.3) have a rather complicated form due to the presence of the Hall–Littlewood polynomials evaluated at a specialization $\mathbf{A} = (\boldsymbol{\alpha}; \boldsymbol{\beta}; \mathbf{Pl}_\gamma)$. Using bivariate dynamics, it is possible to construct sampling algorithms with simpler probabilities as follows.

We will now explain how to sample the measure $\mathcal{HL}_n^{\mathbf{A} \cup \mathbf{B}}$ (\mathbf{A} and \mathbf{B} are arbitrary HL–nonnegative specializations) using the univariate dynamics $\mathbf{Q}_\mathbf{A}$ and an RSK-type bivariate ‘dynamics’ $\mathbf{Q}_{\mathbf{A}; \mathbf{B}}^{(2)}$ (corresponding to functions (W, V) , see §4.4). One can think inductively that we *know* how to sample the measure $\mathcal{HL}_n^\mathbf{A}$ and produce a sampling procedure for the measure $\mathcal{HL}_n^{\mathbf{A} \cup \mathbf{B}}$ with the specialization \mathbf{B} added. In §6.5 below we fully employ such an inductive idea, and also explain how to choose RSK-type bivariate dynamics with nonnegative jump rates (with the help of results of §6.2).

Sampling algorithm 2. *Let ξ_1, \dots, ξ_n be independent identically distributed Bernoulli random variables with values in the two-letter alphabet $\{a, b\}$, and such that for $k = 1, \dots, n$,*

$$\text{Prob}(\xi_k = a) = \frac{p_1(\mathbf{A})}{p_1(\mathbf{A} \cup \mathbf{B})}, \quad \text{Prob}(\xi_k = b) = \frac{p_1(\mathbf{B})}{p_1(\mathbf{A} \cup \mathbf{B})}.$$

Start with a pair of empty Young diagrams $\begin{bmatrix} \lambda(0) \\ \bar{\lambda}(0) \end{bmatrix} = \begin{bmatrix} \emptyset \\ \emptyset \end{bmatrix}$. Clearly, this pair belongs to $\mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$. At each step $k = 1, \dots, n$,

- *If $\xi_k = a$, add to the lower diagram $\bar{\lambda} = \bar{\lambda}(k-1)$ one of the boxes $\bar{\square} \in \mathcal{U}(\bar{\lambda})$ as in Algorithm 1, i.e., with probabilities (6.3). Then add to the upper diagram $\lambda = \lambda(k-1)$ one of the boxes $\square \in \mathcal{U}(\lambda)$ chosen according to the (conditional) ‘probabilities’*

$$\text{‘Prob’}(\lambda \rightarrow \lambda + \square) = V(\lambda, \lambda + \square \mid \bar{\lambda}, \bar{\lambda} + \bar{\square}).$$

- *Else, if $\xi_k = b$, add to the upper Young diagram $\lambda = \lambda(k-1)$ one of the boxes $\square \in \mathcal{U}(\lambda)$ chosen according to ‘probabilities’*

$$\text{‘Prob’}(\lambda \rightarrow \lambda + \square) = \frac{1}{p_1(\mathbf{B})} W(\lambda, \lambda + \square \mid \bar{\lambda}(k-1)),$$

*and let the lower diagram remain the same.*²²

The distribution of the upper Young diagram $\lambda(n)$ is $\mathcal{HL}_n^{\mathbf{A} \cup \mathbf{B}}$.

The fact that this algorithm indeed produces the desired measure $\mathcal{HL}_n^{\mathbf{A} \cup \mathbf{B}}$ readily follows from the definition and properties of bivariate ‘dynamics’ (§4.4).

Remark 6.8. Algorithm 2 allows the specialization \mathbf{A} to be the trivial. In this case, the lower Young diagram always stays empty, and the bivariate ‘dynamics’ $\mathbf{Q}_{\mathbf{A}; \mathbf{B}}^{(2)}$ is reduced to the univariate dynamics $\mathbf{Q}_\mathbf{B}$ (this can be seen from (4.18) because in this case $\mathbb{Y}(\mathbf{A}) = \{\emptyset\}$).

Lemma 6.9. *If in the course of Algorithm 2 one has $\#\{k: \xi_k = b\} = 0$, then the resulting lower and upper Young diagrams coincide, $\bar{\lambda}(n) = \lambda(n)$.*

In this case we say that the diagrams $\bar{\lambda}(n)$ and $\lambda(n)$ are *squashed together*.

²²Thus, the lower Young diagram $\bar{\lambda}(k)$ has a random number of boxes, but $\lambda(k)$ has exactly k boxes.

Proof. Observe that the bivariate ‘dynamics’ lives on the space $\mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$, and for any pair of Young diagrams $\begin{bmatrix} \lambda \\ \bar{\lambda} \end{bmatrix} \in \mathbb{Y}^{(2)}(\mathbf{A}; \mathbf{B})$ one has $\bar{\lambda} \subseteq \lambda$.

Since $\xi_k = a$ for all k , at each k th step of Algorithm 2 we add boxes to both diagrams $\bar{\lambda}(k-1)$ and $\lambda(k-1)$ (i.e., Algorithm 2 is always in case (1)). Arguing by induction, we may assume that $\bar{\lambda}(k-1) = \lambda(k-1)$ (because at step 0 both diagrams are empty and hence coincide). Adding a box \square to the lower Young diagram $\bar{\lambda}(k-1)$ violates the condition $\bar{\lambda} \subseteq \lambda$, and so one must simultaneously add the *same* box $\square = \bar{\square}$ to $\lambda(k-1) = \bar{\lambda}(k-1)$ because it is the only possible way to restore this condition. Indeed, in this case by (4.13) and (4.18) it must be that $V(\lambda, \lambda + \square | \bar{\lambda}, \bar{\lambda} + \bar{\square}) = 1$ (cf. the short-range pushing mechanism (5.13)). Thus, we see that $\bar{\lambda}(k) = \lambda(k)$, and by induction $\bar{\lambda}(n) = \lambda(n)$ as well. \square

One can also readily see that the diagrams $\bar{\lambda}(k)$ and $\lambda(k)$ remain squashed for several initial steps of the sampling algorithm, more precisely, until $\xi_k = b$ for the first time. Moreover, once the diagrams *separate*, they can never get squashed again. Indeed, under an RSK-type bivariate ‘dynamics’ it is impossible to add a box to the lower Young diagram without adding a box to the upper one.

6.4. Towers of Young diagrams, \mathcal{A} -tableaux, and interlacing arrays. Before describing our full sampling algorithm producing the HL-coherent measures $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}$, let us discuss the state space of this algorithm. This space admits several equivalent descriptions explained in Definitions 6.10, 6.11, and 6.13 below.

Definition 6.10 (Towers of Young diagrams). The state space of the sampling algorithm consists of infinite *towers of Young diagrams* $\boldsymbol{\lambda} = \{\lambda^{(a)}\}_{a \in \mathcal{A}}$, where a runs over the alphabet

$$\mathcal{A} := \{1, 2, 3, \dots\} \cup \{\hat{1}, \hat{2}, \hat{3}, \dots\} \cup (0, 1). \quad (6.4)$$

Also denote $\mathcal{A}^{[\alpha]} := \{1, 2, \dots\}$ (*usual letters*), $\mathcal{A}^{[\beta]} := \{\hat{1}, \hat{2}, \dots\}$ (*dual letters*), $\mathcal{A}^{[\mathbf{Pl}]} := (0, 1)$ (*Plancherel letters*). We assume that the alphabet is linearly ordered in some way, say,

$$1 < 2 < 3 < \dots < \hat{1} < \hat{2} < \hat{3} < \dots < \mathcal{A}^{[\mathbf{Pl}]}, \quad (6.5)$$

where points of the continuous segment $\mathcal{A}^{[\mathbf{Pl}]}$ are linearly ordered in the usual way. Young diagrams in the tower should be also ordered (by inclusion) in the same way: for any $a, b \in \mathcal{A}$ with $a < b$, it must be $\lambda^{(a)} \subseteq \lambda^{(b)}$.

A tower of Young diagrams will be called *bounded* if the numbers $\{|\lambda^{(a)}|\}_{a \in \mathcal{A}}$ do not tend to $+\infty$. A bounded towers of Young diagrams possesses a unique maximal (by inclusion) diagram $\lambda^{\max} \in \mathbb{Y}$, and can be interpreted as a certain Young tableaux. (In fact, all towers of Young diagrams produced as outcomes of the full sampling algorithm described in §6.5 below are bounded.)

Definition 6.11 (\mathcal{A} -tableaux). Let λ be a Young diagram. A mapping

$$T: \{\text{boxes of } \lambda\} \rightarrow \mathcal{A}$$

is called an \mathcal{A} -*tableaux of shape* λ (cf. [71], [19]) if the entries $T(\square)$ weakly increase (in the sense of the linear order (6.5) on \mathcal{A}) both along rows and down columns, and, moreover, the following constraints are satisfied (see an example on Fig. 10):

(1) In any column of λ there cannot be two identical letters from $\mathcal{A}^{[\alpha]}$.

- (2) In any row of λ there cannot be two identical letters from $\mathcal{A}^{[\beta]}$.
- (3) Each letter from $\mathcal{A}^{[\mathbf{Pl}]}$ may appear only once in the tableau.

2	2	3	5	0.1
3	5	5	$\hat{7}$	0.34
5	$\hat{5}$			
$\hat{2}$	$\hat{5}$			
$\hat{2}$				

FIGURE 10. An \mathcal{A} -tableau of shape $\lambda = (5, 5, 2, 2, 1)$.

The equivalence between \mathcal{A} -tableaux and bounded towers of Young diagrams is given by

Young diagram $\lambda^{(a)} = \text{Shape inside the } \mathcal{A}\text{-tableau occupied by all letters } b \in \mathcal{A}, b \leq a$, and the maximal diagram λ^{\max} of a tower λ is the same as the shape of the \mathcal{A} -tableau.

Remark 6.12. Note that if an \mathcal{A} -tableau contains only usual letters, then it is simply a semistandard Young tableau, cf. [65, Ch. 7]. If it contains only dual letters, then it is a transpose of a semistandard Young tableau. Finally, if it contains only Plancherel letters, then it is a standard Young tableau.

Definition 6.13 (Interlacing arrays). \mathcal{A} -tableaux (equivalently, bounded towers of Young diagrams) can be represented as interlacing particle arrays as on Fig. 11 (see also §5.2). Such configurations consist of two parts, the lower $\lambda^{[\alpha]}$ and the upper $\lambda^{[\beta \cup \mathbf{Pl}]}$. Levels of $\lambda^{[\alpha]}$ are indexed by usual letters $\in \mathcal{A}^{[\alpha]}$ which enter the corresponding \mathcal{A} -tableau, and levels of $\lambda^{[\beta \cup \mathbf{Pl}]}$ are indexed by dual and Plancherel letters entering this \mathcal{A} -tableau. In $\lambda^{[\alpha]}$, Young diagrams on consecutive levels differ by a *horizontal* strip, and particles correspond to *row* lengths of these Young diagrams. In $\lambda^{[\beta \cup \mathbf{Pl}]}$, consecutive Young diagrams differ by a *vertical* strip, and particles correspond to their *column* lengths. We also add the transpose of the uppermost diagram from $\lambda^{[\alpha]}$ (on Fig. 11 this is $(\lambda^{(5)})'$) to $\lambda^{[\beta \cup \mathbf{Pl}]}$ because the next diagram is obtained from it by adding a vertical strip. Let, by agreement, the number of particles increase by one from level to level inside each of the parts $\lambda^{[\alpha]}$ and $\lambda^{[\beta \cup \mathbf{Pl}]}$. This can always be achieved by appending sequences of row/column lengths by zeroes.

6.5. Full sampling algorithm for $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}$. We are now in a position to describe a sampling algorithm producing the HL-coherent measures $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}$. Let $n = 1, 2, \dots$ and a specialization $(\alpha; \beta; \mathbf{Pl}_\gamma)$ be fixed. According to Remark 3.2, we may assume that

$$p_1(\alpha; \beta; \mathbf{Pl}_\gamma) = \sum_{i \geq 1} \alpha_i + \frac{1}{1-t} \left(\gamma + \sum_{i \geq 1} \beta_i \right) = 1. \quad (6.6)$$

Let $m = m^{\alpha; \beta; \mathbf{Pl}_\gamma}$ be a probability measure on the alphabet \mathcal{A} with

$$m(r) = \alpha_r, \quad m(\hat{r}) = \frac{\beta_r}{1-t}, \quad r = 1, 2, 3, \dots, \quad (6.7)$$

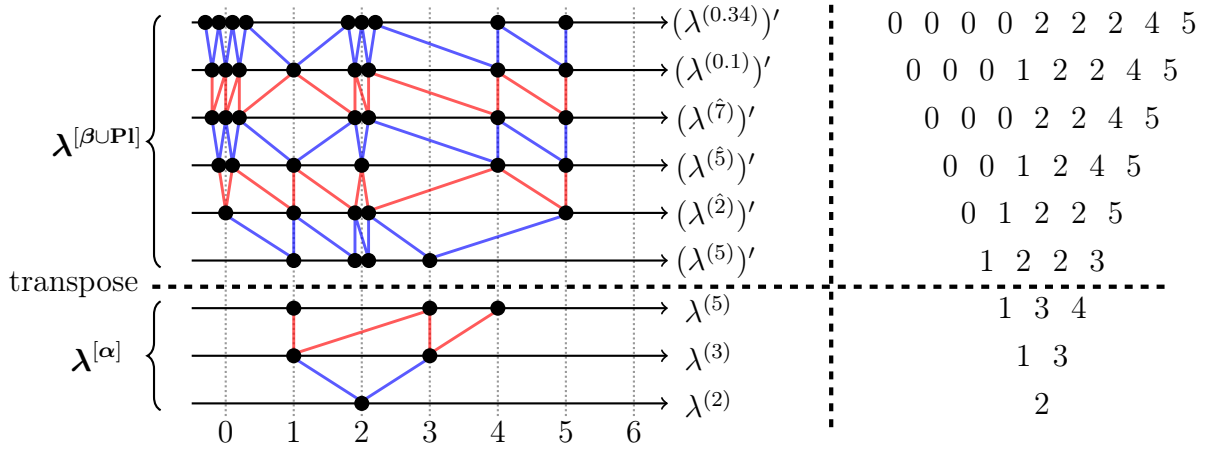


FIGURE 11. A representation of the \mathcal{A} -tableau on Fig. 10 in terms of interlacing arrays; zigzags indicate interlacing of consecutive levels (colors are only to better distinguish the levels). The corresponding array of integers is displayed on the right. Its top level records column lengths of the Young diagram on Fig. 10.

and such that on the segment $\mathcal{A}^{[\text{Pl}]}$, m reduces to a (continuous) uniform measure with $m(\mathcal{A}^{[\text{Pl}]}) = \gamma/(1-t)$. By (6.6), this is indeed a probability measure.

The input of the sampling algorithm is a random word $w = \xi_1 \xi_2 \dots \xi_n$ of length n , where $\xi_i \in \mathcal{A}$ are independent random letters distributed according to $m^{\alpha; \beta; \text{Pl}}_\gamma$. Our sampling algorithm is a combination of the following bivariate dynamics (which are honest Markov processes by Proposition 6.7):

$$\begin{cases} (W_{(\alpha)}^{h=+\infty}, V_{(\alpha)}^{h=+\infty}) & \text{from §5.2 corresponding to a usual letter;} \\ (W_{(\beta)}^{h=1}, V_{(\beta)}^{h=1}) & \text{from §5.3 corresponding to a dual letter;} \\ (W_{\circ}^{\beta; h=1}, V_{\circ}^{\beta; h=1}) & \text{from §5.5 corresponding to a Plancherel letter.} \end{cases} \quad (6.8)$$

It is helpful to keep this list in mind while reading the description of the sampling algorithm below. In particular, conditional probabilities “ V ” above lead to propagation rules described in step (IV).

Sampling algorithm 3. (I) The algorithm starts at step $k = 0$ with an empty interlacing array $\lambda(0)$ containing no particles.

(II) At each step $k = 1, \dots, n$, the modification of the array $\lambda(k-1) \rightarrow \lambda(k)$ begins at level $\lambda^{(\xi_k)}$. If there is no such level $\lambda^{(\xi_k)}$ because the letter ξ_k never appeared before in the random input, then $\lambda^{(\xi_k)}(k-1)$ is created by cloning the previous existing level $\lambda^{(\eta)}$:

- If $k = 1$, then let $\lambda^{(\xi_1)}(0)$ be a single particle at position 0.
- For $k \geq 2$, let η be the maximal letter $\leq \xi_k$ (in terms of the order (6.5) on \mathcal{A}) which has already appeared in the input word, and set $\lambda^{(\xi_k)}(k-1) := \lambda^{(\eta)}(k-1)$.
- After that, add one extra particle at position 0 to all existing levels $\lambda^{(a)}$, $a \geq \xi_k$.

(III) The modification $\lambda^{(\xi_k)}(k-1) \rightarrow \lambda^{(\xi_k)}(k)$ consists in the leftmost free (respectively, the rightmost) particle at level ξ_k moving to the right by one if $\xi_k \in \mathcal{A}^{[\alpha]}$ (respectively, $\xi_k \in \mathcal{A}^{[\beta]} \cup \mathcal{A}^{[\text{Pl}]}$).

(IV) Further modifications of the interlacing array at the same step k consist in propagation of moves through all existing levels $> \xi_k$ (with respect to order (6.5)). This propagation is performed inductively, from the level immediately above ξ_k upwards. Let us fix any two existing levels $d > c \geq \xi_k$ of the interlacing array such that there are no other existing levels between them.

- If $d \in \mathcal{A}^{[\alpha]}$, we argue in terms of row lengths. Suppose that a particle $\lambda_j^{(c)}$ has moved at level c . Then it pushes its first upper right neighbor $\lambda_{\text{next}(j)}^{(d)}$ at level d which is free to move with probability $r_j^{+\infty}(\lambda^{(c)}(k), \lambda^{(d)}(k-1) \mid 0, t)$, or pulls its immediate upper left neighbor $\lambda_{j+1}^{(d)}$ with the complementary probability $1 - r_j^{+\infty}(\lambda^{(c)}(k), \lambda^{(d)}(k-1) \mid 0, t)$, where $r_j^{+\infty}(\cdot, \cdot \mid 0, t)$ is given in Proposition 6.5.
- If $d \in \mathcal{A}^{[\beta]} \cup \mathcal{A}^{[\mathbf{P}]}]$, we will argue in terms of column lengths (this is possible even if $c \in \mathcal{A}^{[\alpha]}$, cf. the “transpose” line on Fig. 11). Suppose that a particle $(\lambda^{(c)})'_j$ has moved at level c . Then it pushes its immediate upper right neighbor $(\lambda^{(d)})'_j$ at level d with probability $r_j^1((\lambda^{(c)}(k))', (\lambda^{(d)}(k-1))' \mid 0, t)$, or pulls its immediate upper left neighbor $(\lambda^{(d)})'_{j+1}$ with the complementary probability $1 - r_j^1((\lambda^{(c)}(k))', (\lambda^{(d)}(k-1))' \mid 0, t)$, where $r_j^1(\cdot, \cdot \mid t, 0)$ is given in Proposition 6.6.

(V) Finally, existing levels $\lambda^{(a)}(k-1)$ for $a < \xi_k$ (in order (6.5)) are not modified at step k .

In (III)–(IV), if the moving particle is blocked from below, then its move is donated to the first free particle to the right of itself (see §5.2 and Fig. 6 in particular). The mandatory short-range pushing mechanism (5.13) is also present in all moves in (IV). Note also that the cloning operation in (II) corresponds to Lemma 6.9.

Since all probabilities involved in the description of Algorithm 3 are nonnegative (by Proposition 6.7), this is indeed an honest randomized sampling procedure.

Theorem 6.14. *Sampling Algorithm 3 indeed produces the HL-coherent measure $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{P} \mathbf{I}_\gamma}$ (as the distribution of the maximal Young diagram $\lambda^{\max}(n)$ at step n).*

Proof. For finitely many nonzero α_i ’s and β_j ’s in the specialization $(\alpha; \beta; \mathbf{P} \mathbf{I}_\gamma)$ this readily follows from Algorithm 2 together with properties of bivariate dynamics (see §4.4 and §5).

When at least one of the sequences $\{\alpha_i\}$ or $\{\beta_j\}$ is infinite, the desired result can be derived via an “algebraic” limit transition. Observe that for each $\lambda \in \mathbb{Y}_n$, the probability weight $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{P} \mathbf{I}_\gamma}(\lambda)$ (3.1) is a power series (with bounded degrees of monomials) which is symmetric in two infinite sets of variables $\{\alpha_i\}_{i=1}^\infty$ and $\{\beta_j\}_{j=1}^\infty$. Indeed, this is because the Hall–Littlewood symmetric functions P_λ are polynomials in the Newton power sums p_1, p_2, \dots , and the specialization $(\alpha; \beta; \mathbf{P} \mathbf{I}_\gamma)$ reduces to the mapping $p_1 \rightarrow 1$, and $p_k \rightarrow p_k(\alpha; \beta; \mathbf{P} \mathbf{I}_\gamma)$ for $k \geq 2$ (see (2.5) with $q = 0$).

Thus, similarly to Remark 2.1, one can also view $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{P} \mathbf{I}_\gamma}(\lambda)$ as a “projective limit” of a sequence of weights

$$\mathcal{H}\mathcal{L}_n^{[N]}(\lambda) = n! P_\lambda(\alpha^{[N]}; \beta^{[N]}; \mathbf{P} \mathbf{I}_\gamma \mid 0, t) Q_\lambda(\mathbf{P} \mathbf{I}_1 \mid 0, t), \quad N \rightarrow \infty,$$

where $\alpha^{[N]} = (\alpha_1, \dots, \alpha_N)$ and $\beta^{[N]} = (\beta_1, \dots, \beta_N)$ are the truncated sequences of parameters. In other words, $\mathcal{H}\mathcal{L}_n^{[N]}(\lambda)$ is the restriction of the desired measure $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{P} \mathbf{I}_\gamma}(\lambda)$ to realizations

of the randomized Algorithm 3 in which there are no letters ξ_i equal to $\alpha_{N+1}, \alpha_{N+2}, \dots$ or $\beta_{N+1}, \beta_{N+2}, \dots$ (note that the weights $\mathcal{H}\mathcal{L}_n^{[N]}(\lambda)$ do not sum to one).

One can readily check that our Algorithm 3 is compatible with that “projective limit” as well. This completes the proof for all specializations $(\alpha; \beta; \mathbf{Pl}_\gamma)$. \square

Let us make several comments on Sampling Algorithm 3.

Remark 6.15. It is possible to reword the above description purely in terms of operations on \mathcal{A} -tableaux. Namely, at each step k the new letter $\xi_k \in \mathcal{A}$ distributed according to the measure (6.7) is *inserted* into the existing \mathcal{A} -tableau. The insertion always starts from the first column of the tableau. All following elementary *bumping/shifting* operations in the course of the insertion are performed at random according to probabilities in step (IV) of Algorithm 3 (this also depends on the type of the letter ξ_k). See also [16, §7.2] for more detail on how the setting of interlacing arrays can be translated into the language of Young tableaux.

Remark 6.16. As can be readily seen from the construction of the full sampling algorithm, other choices of a linear order on \mathcal{A} lead to other (different) algorithms which, however, sample *the same* HL-coherent measure. Equivalently, permuting the probability weights assigned by the measure m (e.g., setting $m^{\alpha; \beta; \mathbf{Pl}_\gamma}(i) = \alpha_{\sigma(i)}$ for any permutation σ of natural numbers) also amounts to sampling of the same HL-coherent measure. We will utilize such different orderings below in §7.

Remark 6.17. Notice that if letters ξ_i in the random input word w appear in continuous time according to certain independent Poisson clocks, then after time $\tau \geq 0$, Algorithm 3 produces the poissonized version of the HL-coherent measure, i.e., the Macdonald measure $\mathcal{MM}_\tau^{\alpha; \beta; \mathbf{Pl}_\gamma}$ (§3.2) with $q = 0$. More precisely, letters r and \hat{r} ($r = 1, 2, \dots$) should appear at rates α_r and $\frac{\beta_r}{1-t}$, respectively, and letters from $\mathcal{A}^{[\mathbf{Pl}]}$ should appear at total rate $\frac{\gamma}{1-t}$. Note that each letter from $\mathcal{A}^{[\mathbf{Pl}]}$ is almost surely new, i.e., it has not appeared in the word before (cf. (6.7)).

Remark 6.18. In the course of the sampling algorithm, we construct a sequence of random Young diagrams $\lambda(n)$, $|\lambda(n)| = n$, on the same probability space. The marginal distribution of each $\lambda(n)$ is $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}$. It can be readily seen that the joint distribution of this sequence of random Young diagrams reflects the coherency property of the measures $\mathcal{H}\mathcal{L}_n^{\alpha; \beta; \mathbf{Pl}_\gamma}$ (see §5.4). When $t = q^{-1} = p^{-d}$ is the inverse of a prime power, this “big” probability space corresponds to the distribution of the infinite uni-uppertriangular random matrix over the finite field F_q , see §1.1 and Remark 1.3 in particular.

Remark 6.19. In the Schur case, i.e., for $t = 0$, all randomized insertion steps become *deterministic* (see also [16, §7] for more discussion), and one arrives at (a column version of) the generalization of the Robinson–Schensted–Knuth (RSK) insertion introduced in [71] (and also employed in, e.g., [19], [64]). In fact, it is possible to obtain the algorithm of [71] itself by taking a suitable order on \mathcal{A} (cf. (6.5)), and also suitable other indices h in (6.8).

6.6. An example of the randomized insertion. Let us demonstrate the randomized insertion of the letter $\xi = 3$ into the \mathcal{A} -tableau on Fig. 10. We will use integer arrays as on Fig. 11, right. The letter 3 is already present in the tableau, so no cloning of levels will occur. The initial insertion means that the leftmost particle at level $\lambda^{(3)}$ jumps to the right by one (observe that this particle is free to jump, so no move donation occurs). Then, with probability

$r_2^{+\infty}((3, 2), (4, 3, 1) | 0, t) = 1 - t$, this particle pushes the first free particle to the right on the next level. With the complementary probability t it pulls its upper left neighbor. Thus, after propagation of the move to the level $\lambda^{(5)}$, the random configuration on levels $\lambda^{(2)} \prec_h \lambda^{(3)} \prec_h \lambda^{(5)}$ looks as follows (here and below “w.p.” stands for “with probability”):

$$\begin{array}{|c|c|c|} \hline \lambda^{(5)} & 1 & 3 \boxed{5} \\ \hline \lambda^{(3)} & \boxed{2} & 3 \\ \hline \lambda^{(2)} & & 2 \\ \hline \end{array} \quad \text{w.p. } 1 - t, \quad \begin{array}{|c|c|c|} \hline \lambda^{(5)} & \boxed{2} & 3 \ 4 \\ \hline \lambda^{(3)} & \boxed{2} & 3 \\ \hline \lambda^{(2)} & & 2 \\ \hline \end{array} \quad \text{w.p. } t. \quad (6.9)$$

In the first case, the propagation to several next levels is deterministic due to the short-range pushing, cf. (5.13). Note that here we had to add one more (leftmost) particle to each level in the upper part $\lambda^{[\beta \cup \mathbf{PI}]}$. The resulting distribution in the first case looks as follows:

$$\begin{array}{|c|c|c|} \hline \begin{array}{cccccccc} 0 & 0 & 0 & 0 & \boxed{1} & 2 & 2 & 2 & 4 & 5 \\ 0 & 0 & 0 & \boxed{1} & 1 & 2 & 2 & 4 & 5 \\ 0 & 0 & 0 & \boxed{1} & 2 & 2 & 4 & 5 \\ 0 & 0 & \boxed{1} & 1 & 2 & 4 & 5 \\ 0 & \boxed{1} & 1 & 2 & 2 & 5 \\ \boxed{1} & 1 & 2 & 2 & 3 \\ \hline & 1 & 3 & \boxed{5} \\ & \boxed{2} & 3 \\ & 2 \end{array} & \text{w.p. } (1-t) \frac{1}{1+t}, & \begin{array}{cccccccc} 0 & 0 & 0 & 0 & \boxed{1} & 2 & 2 & 2 & 4 & 5 \\ 0 & 0 & 0 & 0 & \boxed{2} & 2 & 2 & 4 & 5 \\ 0 & 0 & 0 & \boxed{1} & 2 & 2 & 4 & 5 \\ 0 & 0 & \boxed{1} & 1 & 2 & 4 & 5 \\ 0 & \boxed{1} & 1 & 2 & 2 & 5 \\ \boxed{1} & 1 & 2 & 2 & 3 \\ \hline & 1 & 3 & \boxed{5} \\ & \boxed{2} & 3 \\ & 2 \end{array} & \text{w.p. } (1-t) \frac{t}{1+t}. \\ \hline \end{array} \quad (6.10)$$

Indeed, this is because $r_5^1((5, 4, 2, 2, 1, 0, 0, 0), (5, 4, 2, 2, 1, 0, 0, 0, 0) | t, 0) = \frac{t}{1+t}$ (propagation from $(\lambda^{(\hat{7})})'$ to $(\lambda^{(0.1)})'$, cf. Fig. 11), and $r_5^1((5, 4, 2, 2, 2, 0, 0, 0, 0), (5, 4, 2, 2, 2, 0, 0, 0, 0) | t, 0) = 0$ (propagation from $(\lambda^{(0.1)})'$ to $(\lambda^{(0.34)})'$).

Remark 6.20. In fact, in the upper part $\lambda^{[\beta \cup \mathbf{PI}]}$ of the array there are no donations of jumps or pushes, see [16, §8.2.1].

In the second case, the resulting distribution looks as follows:

$$\begin{array}{|c|c|c|} \hline \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 2 & 2 & \boxed{3} & 4 & 5 \\ 0 & 0 & 0 & 0 & 2 & \boxed{3} & 4 & 5 \\ 0 & 0 & 0 & 2 & \boxed{3} & 4 & 5 \\ 0 & 0 & 1 & \boxed{3} & 4 & 5 \\ 0 & 1 & 2 & \boxed{3} & 5 \\ \boxed{1} & 2 & \boxed{3} & 3 \\ \hline & \boxed{2} & 3 & 4 \\ & \boxed{2} & 3 \\ & 2 \end{array} & \text{w.p. } t \frac{1+t}{1+t+t^2}, & \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 2 & 2 & 2 & \boxed{5} & 5 \\ 0 & 0 & 0 & 0 & 2 & 2 & \boxed{5} & 5 \\ 0 & 0 & 0 & 2 & 2 & \boxed{5} & 5 \\ 0 & 0 & 1 & 2 & \boxed{5} & 5 \\ 0 & 1 & 2 & \boxed{3} & 5 \\ \boxed{1} & 2 & \boxed{3} & 3 \\ \hline & \boxed{2} & 3 & 4 \\ & \boxed{2} & 3 \\ & 2 \end{array} & \text{w.p. } t \frac{t^2}{1+t+t^2}. \\ \hline \end{array} \quad (6.11)$$

Indeed, one can check that $r_2^1((5, 3, 2, 1, 0), (5, 4, 2, 1, 0, 0) | t, 0) = \frac{t^2}{1+t+t^2}$ (propagation from $(\lambda^{(\hat{2})})'$ to $(\lambda^{(\hat{5})})'$, cf. Fig. 11), and propagation to all further levels is deterministic.

Thus, we see that the result of the insertion is a random \mathcal{A} -tableau described in (6.10)–(6.11).

7. PROOF OF THE LAW OF LARGE NUMBERS

This section is devoted to the proof of the main result of the present paper (see §1.2 in the Introduction). Throughout the section we assume that the Macdonald parameter q is set to zero. Fix two sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ and $\beta = (\beta_1, \beta_2, \dots)$ such that $p_1(\alpha; \beta) = \sum_{i=1}^{\infty} \alpha_i + \frac{1}{1-t} \sum_{i=1}^{\infty} \beta_i = 1$ (cf. (6.6)). That is, we will consider a HL-nonnegative specialization $(\alpha; \beta; \mathbf{Pl}_\gamma) = (\alpha; \beta; \mathbf{Pl}_0)$ with the Plancherel parameter γ set to zero.

Theorem 7.1. *Let for each $n \geq 1$, $\lambda(n)$ be a random Young diagram with n boxes distributed according to the HL-coherent measure $\mathcal{HL}_n^{\alpha; \beta; \mathbf{Pl}_0}$ (defined in §3.1; we assume that the parameters satisfy (6.6)). The following Law of Large Numbers holds:*

$$\frac{\lambda_i(n)}{n} \rightarrow \alpha_i, \quad \frac{\lambda'_i(n)}{n} \rightarrow \frac{\beta_i}{1-t}, \quad n \rightarrow \infty, \quad (7.1)$$

where λ'_i are the column lengths of λ (i.e., row lengths of the transposed diagram λ).

The convergence in (7.1) is almost sure with respect to the probability space carrying all random Young diagrams $\lambda(n)$, $n = 1, 2, 3, \dots$ (cf. Remarks 6.18 and 1.3).

See §1.2 in the Introduction for further conjectures related to Theorem 7.1.

7.1. Strategy of the proof. Let us briefly outline our strategy of the proof of Theorem 7.1. Assume first that both sequences α and β are finite,

$$\alpha = \{\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_a\}, \quad \beta = \{\beta_1 \geq \beta_2 \geq \dots \geq \beta_b\}.$$

Alphabet \mathcal{A} (6.4) then reduces to

$$\mathcal{A} = \{1, 2, \dots, a\} \cup \{\hat{1}, \hat{2}, \dots, \hat{b}\}.$$

Let us slightly modify (with the help of Remark 6.16) the probability distribution $m = m^{\alpha; \beta}$ (6.7) on \mathcal{A} such that

$$m^{\alpha; \beta}(i) = \alpha_i, \quad i = 1, \dots, a; \quad m^{\alpha; \beta}(\hat{j}) = \frac{\beta_{b+1-j}}{1-t}, \quad j = 1, \dots, b, \quad (7.2)$$

so that $m(1) \geq m(2) \geq \dots \geq m(a)$, and $m(\hat{1}) \leq m(\hat{2}) \leq \dots \leq m(\hat{b})$.

7.1.1. Row lengths. To deal with row lengths, we order the alphabet \mathcal{A} as

$$1 < 2 < \dots < a < \hat{1} < \hat{2} < \dots < \hat{b}$$

(this ordering coincides with the one considered in §6). Then the lower part $\lambda^{[\alpha]}$ of the array on Fig. 11 consists of a finite number of interlacing particles, namely, $\{\lambda_i^{(m)}\}$, where $m = 1, \dots, a$, $j = 1, \dots, m$. To establish the row part of Theorem 7.1 (i.e., the convergence $\lambda_j(n)/n \rightarrow \alpha_j$), it suffices to show that the coordinates of the particles at the top a th level behave as $\lambda_j^{(a)}(n)/n \rightarrow \alpha_j$ as $n \rightarrow \infty$, $j = 1, \dots, a$. Indeed, recall the bijection of arrays with \mathcal{A} -tableaux (§6.4), and observe that in each row of an \mathcal{A} -tableau (as on Fig. 10) there can be at most one letter of the form \hat{i} , where $i = 1, \dots, b$.

Next, note that the evolution of the particle coordinates $\lambda_j^{(a)}(n)$ depends only on the particles in the lower part $\lambda^{[\alpha]}$ of the array. Thus, we can freely assume that all the parameters β_j of our specialization are zero, and

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_a \geq 0, \quad \sum_{i=1}^a \alpha_i = 1. \quad (7.3)$$

This is the setup which we are going to apply to prove the row part of Theorem 7.1 (in the case of both sequences α and β being finite).

Our strategy of the proof relies on the observation that when (in the course of the sampling algorithm) each particle is far enough from its upper left neighbor, a move of $\lambda_j^{(m)}$ triggers the move of its upper right neighbor $\lambda_j^{(m+1)}$, which in turn pushes $\lambda_j^{(m+2)}$, and so on. Our choice of speeds of independently jumping particles $\lambda_m^{(m)}$ (i.e., the bottommost particle $\lambda_1^{(1)}$ is the fastest one) informally suggests that the particles will be in that generic situation most of the time, and the whole array will asymptotically look like on Fig. 12. That is, particles $\lambda_j^{(m)}$ with fixed j and m running over $j, j+1, \dots, a$ will have horizontal coordinate $\sim \alpha_j n$ (plus random fluctuations of order \sqrt{n} , cf. Remark 7.7). It is helpful to keep this observation in mind while reading the rigorous arguments below in this section.

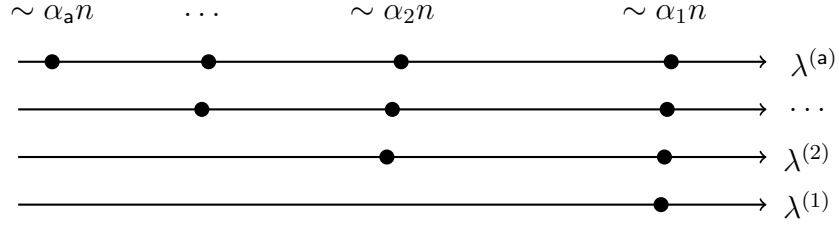


FIGURE 12. Clusters of particles under a specialization $\alpha = (\alpha_1, \dots, \alpha_a)$ into a usual variables.

7.1.2. *Column lengths.* For the purpose of dealing with column lengths, we reorder \mathcal{A} as

$$\hat{1} < \hat{2} < \dots < \hat{b} < 1 < 2 < \dots < a$$

(this is allowed by Remark 6.16).

In this case by a similar argument we see that it also suffices to consider only the case when all α_j 's are zero, and the specialization has the parameters

$$\beta_1 \geq \beta_2 \geq \dots \geq \beta_b, \quad \sum_{i=1}^b \beta_i = (1 - t). \quad (7.4)$$

Then in the array λ there is a finite number of interlacing particles $(\lambda_j^{(\hat{m})})'$, $m = 1, \dots, b$, $j = 1, \dots, m$, on the first b levels. To shorten the notation, denote their coordinates by

$\tau_j^{(m)} := (\lambda_j^{(m)})'$.²³ To establish the column part of Theorem 7.1, we need to show that the top row particles behave as $\tau_j^{(b)}(n)/n \rightarrow \beta_j/(1-t)$ as $n \rightarrow \infty$, $j = 1, \dots, b$.

Our choice of speeds of independently jumping particles $\tau_1^{(m)}$ (i.e., the bottommost particle $\tau_1^{(1)}$ is the slowest one) informally suggests that our interlacing particles behave as on Fig. 13 (plus random fluctuations of order \sqrt{n} , cf. Remark 7.7). Indeed, when the particles are sufficiently far from each other, a move of any particle $\lambda_j^{(m)}$ will trigger the move of its upper left neighbor $\lambda_{j+1}^{(m+1)}$. It is also helpful to keep this observation in mind while reading the rigorous arguments below.

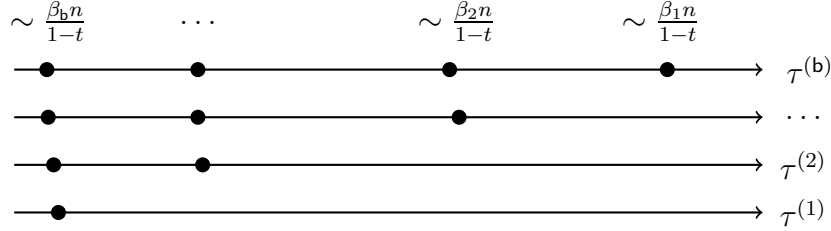


FIGURE 13. Clusters of particles under a specialization $\beta = (\beta_1, \dots, \beta_b)$ into b dual variables.

Remark 7.2. The evolution of $\{\tau_1^{(m)}\}_{m=1}^b$ becomes the q -PushTASEP [16], [20] after the renaming t by q and considering the evolution in continuous time. Thus, Theorem 7.1 that we are proving implies existence of asymptotic speeds of particles under the q -PushTASEP.

7.1.3. Outline of the section. In §7.2 we define certain simple Markov dynamics which are used in coupling arguments in the case when both sequences of parameters α and β are finite. Proofs in this case are presented in §7.3 (row part) and §7.4 (column part). With the help of an additional argument, the case when sequences α and β can be infinite is reduced to the case when both these sequences are finite. We deal with this reduction in §7.5 below.

7.2. Auxiliary dynamics and estimates. Let $m(n)$ ($n \geq 0$ — discrete time) be a Markov chain with state space $\mathbb{Z}_{\geq 0}$ depending on parameters $p_+ \geq p_- > 0$ and $p_0 > 0$ with one-step transitions defined as

$$m(n+1) = \begin{cases} m(n) + 1, & \text{with probability } p_+; \\ m(n) - 1, & \text{with probability } p_-; \\ m(n), & \text{with probability } 1 - p_+ - p_- \end{cases}$$

if $m(n) \neq 0$, and

$$m(n+1) = \begin{cases} 1, & \text{with probability } p_0; \\ 0, & \text{with probability } 1 - p_0. \end{cases}$$

²³In fact, these particles evolve according to Dynamics 8 in [16], up to renaming t by q and considering the evolution in continuous time, cf. Remark 6.17.

For any $c \in (0, 1)$, define

$$\mathbf{w}(n) := \sum_{i=0}^n \mathbf{1}_{\mathbf{m}(i)=0}, \quad \phi(n) := \sum_{i=0}^n c^{\mathbf{m}(i)}. \quad (7.5)$$

Lemma 7.3. *As $n \rightarrow \infty$,*

$$\mathbb{E} \mathbf{w}(n) = O(\sqrt{n}), \quad \mathbb{E} \phi(n) = O(\sqrt{n}).$$

Proof. Clearly, it suffices to establish the statement of the lemma for the case $p_+ = p_-$. Let $\{\nu_i\}_{i \geq 1}$ be independent random variables with distribution $p_- \delta_{-1} + (1 - 2p_-) \delta_0 + p_- \delta_1$ (here δ means the Dirac probability distribution at a point). Denote

$$\mathbf{m}'(k) := |\nu_1 + \nu_2 + \dots + \nu_k|, \quad \mathbf{w}'_r(k) := \sum_{i=0}^k \mathbf{1}_{\mathbf{m}(i)=r},$$

where $r = 0, 1, 2, \dots$

It can be readily shown (for example, using the standard Central Limit Theorem) that

$$P(\mathbf{m}'(k) = r) \leq \frac{\text{const}}{\sqrt{k}}$$

for some *const* not depending on k . Therefore,

$$\mathbb{E} \mathbf{w}'_r(n) = \sum_{k=0}^n P(\mathbf{m}'(k) = r) \leq \sum_{k=0}^n \frac{\text{const}}{\sqrt{k}} \leq \text{const} \sqrt{n}.$$

The chain $\mathbf{m}(n)$ differs from $\mathbf{m}'(n)$ only by transition probabilities at zero. Therefore, because on average the chain $\mathbf{m}(n)$ spends time $1/p_0$ at zero, we have

$$\mathbb{E} \sum_{i=0}^n \mathbf{1}_{\mathbf{m}(i)=r} \leq \frac{1}{p_0} \mathbb{E} \mathbf{w}'_r(n) \leq \text{const} \sqrt{n}$$

for any $r = 0, 1, \dots$. This readily implies the desired estimates. \square

Now let us introduce two auxiliary many-particle systems on \mathbb{Z} . The first of the systems is (the discrete-time version of, cf. Remark 6.17) the well-known Totally Asymmetric Simple Exclusion Process (TASEP) with a finite number $a = 1, 2, \dots$ of particles, and with particle-dependent speeds $\{\alpha_i\}_{i=1}^a$ satisfying (7.3).

Let us denote positions of the particles in that process by $\mathbf{T}_1(n) \geq \mathbf{T}_2(n) \geq \dots \geq \mathbf{T}_a(n)$ (n is the discrete time). The dynamics preserves this ordering. The evolution of this TASEP goes as follows. Initially, $\mathbf{T}_1(0) = \mathbf{T}_2(0) = \dots = \mathbf{T}_a(0) = 0$. At each moment of the discrete time, exactly one of the particles receives a “jumping signal”; the probability that this is \mathbf{T}_i is equal to α_i (independently of previous “signals”). Then, if $\mathbf{T}_i(n) < \mathbf{T}_{i-1}(n)$, the coordinate of \mathbf{T}_i increases by one. Otherwise, if $\mathbf{T}_i(n) = \mathbf{T}_{i-1}(n)$, then no jump occurs (so the particle \mathbf{T}_i can be blocked by the next one \mathbf{T}_{i-1}).

Proposition 7.4. *The dynamics $\{\mathbf{T}_i(n)\}_{i=1}^a$ satisfies the following Law of Large Numbers:*

$$\frac{\mathbf{T}_i(n)}{n} \rightarrow \alpha_i, \quad n \rightarrow \infty, \quad \text{almost surely.}$$

Proof. This fact is well-known. However, we were not able to find an exact reference in the literature. For the sake of completeness we provide one of many possible explanations.

The TASEP $\{\mathbb{T}_i(n)\}_{i=1}^a$ can be identified with the dynamics of the leftmost particles in an interlacing array under the column RSK insertion process, e.g., see [16, §7.1.3]. (This fact can be traced back to [42], [3], [53], [54], see also [10].) Therefore, $\mathbb{T}_i(n)$ can be identified with the last row λ_i of a Young diagram λ with $\ell(\lambda) \leq i$ distributed as follows. First, let k be a binomial random variable with distribution

$$\text{Prob}(k) = \binom{n}{k} (\alpha_1 + \dots + \alpha_i)^k (\alpha_{i+1} + \dots + \alpha_a)^{n-k}$$

(recall the condition (7.3) on α_j 's). Then, given k , let λ be a Young diagram with the distribution

$$\text{Prob}(\lambda) := \frac{k!}{(\alpha_1 + \dots + \alpha_i)^k} s_\lambda(\alpha_1, \dots, \alpha_i) s_\lambda(\mathbf{P}\mathbf{1}_1), \quad |\lambda| = k, \quad \ell(\lambda) \leq i \quad (7.6)$$

(cf. (1.9), note also that this distribution is a particular $q = t$ case of the coherent measures discussed in §3.1).

The Law of Large Numbers for the measures (7.6) (for nonrandom growing k) was established in [68], stating that

$$\frac{\lambda_i(k)}{k} \rightarrow \frac{\alpha_i}{\alpha_1 + \dots + \alpha_i}, \quad k \rightarrow \infty.$$

Since k satisfies the classical Law of Large Numbers, $k \approx (\alpha_1 + \dots + \alpha_i)n$, a standard argument shows that the desired statement holds. \square

Let us introduce the second dynamics which has $\mathbf{b} \geq 1$ particles on \mathbb{Z} and depends on our Hall–Littlewood parameter $0 < t < 1$, as well as on parameters $\{\beta_j\}_{j=1}^{\mathbf{b}}$ satisfying (7.4). Let us denote positions of the particles in our second process by $\mathbf{Q}_1(n) \leq \mathbf{Q}_2(n) \leq \dots \leq \mathbf{Q}_{\mathbf{b}}(n)$ (n is the discrete time). The dynamics preserves this ordering. The evolution of this system is described as follows. Initially, all particles start at zero. Next, at each moment of the discrete time, exactly one of the particles, namely, \mathbf{Q}_i with probability $\beta_{\mathbf{b}+1-j}/(1-t)$, receives a “jumping signal”. The particle \mathbf{Q}_i which received this signal jumps to the right by one (it cannot be blocked). After that, with probability $t^{\mathbf{Q}_{i+1}(n)-\mathbf{Q}_i(n)}$, all particles $\mathbf{Q}_{i+1}, \mathbf{Q}_{i+2}, \mathbf{Q}_{i+3}, \dots$ also move to the right by one. With the complementary probability $1 - t^{\mathbf{Q}_{i+1}(n)-\mathbf{Q}_i(n)}$, no particles other than \mathbf{Q}_i (which has already jumped) move. Note that if $\mathbf{Q}_{i+1}(n) = \mathbf{Q}_i(n)$, then the pushing probability reduces to 1.

We have chosen the speeds of independently jumping particles in this dynamics so that the first particle \mathbf{Q}_1 is the slowest one, cf. §7.1.2.

Remark 7.5. Note the difference between the Markov dynamics $\{\mathbf{Q}_j(n)\}_{j=1}^{\mathbf{b}}$ and the Markov evolution of the particles $\{\tau_1^{(m)}\}_{m=1}^{\mathbf{b}}$ from §7.1.2. In the former dynamics, if a long-range push happens (with probability $t^{\mathbf{Q}_{i+1}(n)-\mathbf{Q}_i(n)}$), then all particles with indices $i+1, i+2, \dots$ move. In the latter dynamics, a long-range push is applied only to the $(i+1)$ -st particle, and further pushes (of particles $i+2, i+3, \dots$) happen under an additional randomness.

Proposition 7.6. *The dynamics $\{Q_j(n)\}_{j=1}^b$ satisfies the following Law of Large Numbers:*

$$\frac{Q_j(n)}{n} \rightarrow \frac{\beta_{b+1-j}}{1-t}, \quad n \rightarrow \infty, \quad \text{almost surely.}$$

Proof. Denote $d_i(n) := Q_{i+1}(n) - Q_i(n)$. This quantity increases when Q_{i+1} jumps independently, and decreases when Q_i jumps independently and does not push Q_{i+1} . A natural coupling with the process $m(n)$ from Lemma 7.3 implies

$$\sum_{k=0}^n t^{d_i(k)} \leq \text{const} \sqrt{n}.$$

Therefore, the expectation of the number of times (before time n) when an independent jump of Q_i leads to a move of Q_{i+1} can be estimated by $\text{const} \sqrt{n}$. We see that the evolution of each particle Q_i is affected by pushes $\leq \text{const} \sqrt{n}$ times, so the asymptotic speed of this particle is determined by its independent jumps. This concludes the proof. \square

It is helpful to look at Fig. 12 and 13 in connection with Propositions 7.4 and 7.6, respectively.

Remark 7.7. When $\gamma = 0$ and the remaining parameters are distinct (when they are positive), i.e., $\alpha_1 > \alpha_2 > \dots$ and $\beta_1 > \beta_2 > \dots$, the estimates of this subsection can be readily improved from $O(\sqrt{n})$ to $O(1)$. In this case, arguments of §§7.3–7.5 should give a Central Limit Theorem (Conjecture 1.8), which would follow from the corresponding Central Limit Theorem for the independent random letters in the input word $w = \xi_1 \xi_2 \dots$ (§6.5). This explains the nature of the covariance matrix in Conjecture 1.8.

7.3. Finitely many usual variables, and row lengths. Here we will consider the evolution of an interlacing particle array $\{\lambda_j^{(m)}(n)\}$, where $m = 1, \dots, a$, $j = 1, \dots, m$, which depends on parameters $\{\alpha_i\}_{i=1}^a$ satisfying (7.3) (see §7.1.1 and Fig. 12). We can assume that all β_j 's are zero, see §7.1. Under this evolution, only the leftmost particles $\lambda_m^{(m)}$, $m = 1, \dots, a$, can jump independently. Let $N_m(n)$ be the number of independent jumps performed by the particle $\lambda_m^{(m)}$. Note that the bottommost particle $\lambda_1^{(1)}$ is the fastest one.

The goal of this section is to prove Proposition 7.9.

Lemma 7.8. *The following convergence holds for every $i = 1, \dots, a$:*

$$\frac{\lambda_i^{(i)}(n)}{n} \rightarrow m^{\alpha;\beta}(i) = \alpha_i, \quad n \rightarrow \infty, \quad \text{almost surely.}$$

Proof. The key point in the proof is to bound the quantities $\lambda_i^{(i)}(n)$ from below, which is done by means of their coupling with the dynamics $\{T_i(n)\}_{i=1}^a$ from §7.2. We will put both dynamics $\{\lambda_i^{(m)}(n)\}$ and $\{T_i(n)\}$ on the same probability space Ω_n (with n being arbitrary), such that

$$\lambda_i^{(i)}(n) \geq T_i(n) \quad \text{everywhere on } \Omega_n. \quad (7.7)$$

We are assuming that both dynamics depend on the same parameters $\{\alpha_i\}_{i=1}^a$ satisfying (7.3).

Let us begin with defining the space Ω_n for the dynamics on interlacing triangular arrays $\{\lambda_i^{(m)}(n)\}$. Initially, Ω_0 contains one point, and all $\lambda_i^{(m)}(0)$'s are equal to zero. The passage from Ω_n to Ω_{n+1} amounts to splitting every element of Ω_n into a parts of relative probability

weights $\alpha_1, \dots, \alpha_a$; the i -th such part corresponds to the particle $\lambda_i^{(i)}$ jumping independently. Furthermore, one has to account for probabilities of subsequent pushes (see §6.5).

Thus, every element of Ω_{n+1} can be described as a word $w = \xi_1 \dots \xi_{n+1}$, $\xi_i \in \{1, 2, \dots, a\}$ (this is the input word of the sampling algorithm, cf. §6.5), plus the whole history of the dynamics of the interlacing array

$$\{\lambda_i^{(m)}(k) : m = 1, \dots, a; i = 1, \dots, m; k = 1, \dots, n+1\}.$$

It is clear how the dynamics on interlacing arrays assigns a probability weight to each such element of Ω_{n+1} .

The dynamics $\{\mathsf{T}_i(n)\}$ can also be put onto probability space Ω_n in a rather straightforward way. Namely, projecting an element of Ω_n to the word $w = \xi_1 \dots \xi_n$ in the alphabet $\{1, 2, \dots, a\}$, we associate to each new incoming letter ξ_j the independent jump of the particle T_{ξ_j} in the TASEP. In other words, particles $\lambda_i^{(i)}$ and T_i receive “jump signals” simultaneously.

Let us now establish (7.7). Initially, at time $n = 0$, (7.7) clearly holds. By induction on n , assume that these inequalities hold for each element of Ω_n . Let the new letter be $\xi_{n+1} = k$. Then

- If $\lambda_k^{(k)}(n) > \mathsf{T}_k(n)$ at step n , then at step $n+1$ all the inequalities continue to hold. Indeed, at the next step each of the coordinates $\lambda_k^{(k)}$ and T_k can potentially increase by 1, and no other coordinates of the form $\lambda_j^{(j)}$ and T_j , $j \neq k$ can increase.²⁴
- If $\lambda_k^{(k)}(n) = \mathsf{T}_k(n)$ and $\mathsf{T}_{k-1}(n) > \mathsf{T}_k(n)$, then $\lambda_k^{(k)}(n) < \lambda_{k-1}^{(k-1)}(n)$ by the induction hypothesis. Thus, both particles $\lambda_k^{(k)}$ and T_k are not blocked and thus jump, and the inequalities continue to hold (since no other particles $\lambda_j^{(j)}$ and T_j , $j \neq k$, move).
- If $\lambda_k^{(k)}(n) = \mathsf{T}_k(n)$ and $\mathsf{T}_{k-1}(n) = \mathsf{T}_k(n)$, then the particles T_j , $j = 1, \dots, a$, do not move at the next step. On the other hand, $\lambda_k^{(k)}$ can potentially jump (if it is not blocked), while other particles $\lambda_j^{(j)}$, $j \neq k$, will not move. Thus, the inequalities continue to hold.

This argument completes the proof of (7.7).

Now let us finish the proof of the lemma. From (7.7) and Proposition 7.4 it follows that

$$\liminf_{n \rightarrow \infty} \frac{\lambda_i^{(i)}(n)}{n} \geq \alpha_i, \quad \text{almost surely}$$

for every $i = 1, \dots, a$. Moreover, $\lambda_i^{(a)}(n) \geq \lambda_i^{(i)}(n)$, and $\sum_{i=1}^n \lambda_i^{(a)}(n) = n$. By an elementary contradiction argument, this implies the claim of the lemma. \square

Proposition 7.9. *The following Law of Large Numbers holds for every $m = 1, \dots, a$ and every $i = 1, \dots, m$:*

$$\frac{\lambda_i^{(m)}(n)}{n} \rightarrow m^{\alpha_i \beta}(i) = \alpha_i, \quad n \rightarrow \infty, \quad \text{almost surely.} \quad (7.8)$$

²⁴Of course, particles in the bulk of the interlacing array (i.e., all particles except the leftmost ones $\lambda_i^{(i)}$) will move in some way, but this cannot break the desired inequalities. This remark applies to other two cases as well.

Proof. We argue by induction on $i = m, m-1, \dots, 1$, and the case $i = m$, i.e., the convergence of $\lambda_m^{(m)}(n)/n$, follows from Lemma 7.8. Now assume that we have established the convergence of $\lambda_i^{(m)}(n)/n$ for all $i = r+1, \dots, m$, and let us prove that $\lambda_r^{(m)}(n)/n \rightarrow \alpha_r$.

Clearly, $\lambda_i^{(m)}(n) \geq \lambda_i^{(i)}(n)$ due to interlacing. Therefore, by Lemma 7.8,

$$\liminf_{n \rightarrow \infty} \frac{\lambda_i^{(m)}(n)}{n} \geq \alpha_i \quad \text{almost sure.} \quad (7.9)$$

On the other hand, we know that

$$\sum_{i=1}^m \lambda_i^{(m)}(n) = \sum_{i=1}^m N_i(n),$$

(because at the n th step of the sampling algorithm the Young tableau has exactly n boxes, cf. §6). From the classical Law of Large Numbers for independent random variables we have

$$\frac{N_i(n)}{n} \rightarrow \alpha_i, \quad n \rightarrow \infty \quad \text{almost surely.}$$

This implies

$$\begin{aligned} & \alpha_m + \dots + \alpha_{r+1} + \limsup_{n \rightarrow \infty} \frac{\lambda_r^{(m)}(n)}{n} \\ &= \limsup_{n \rightarrow \infty} \frac{\lambda_m^{(m)}(n) + \lambda_{m-1}^{(m)}(n) + \dots + \lambda_r^{(m)}(n)}{n} \\ &= \limsup_{n \rightarrow \infty} \frac{(N_1(n) + \dots + N_m(n)) - \lambda_1^{(m)}(n) - \lambda_2^{(m)}(n) - \dots - \lambda_{r-1}^{(m)}(n)}{n} \\ &= (\alpha_1 + \dots + \alpha_m) - \liminf_{n \rightarrow \infty} \frac{\lambda_1^{(m)}(n) + \lambda_2^{(m)}(n) + \dots + \lambda_{r-1}^{(m)}(n)}{n} \leq (\alpha_1 + \dots + \alpha_m) - \alpha_1 - \dots - \alpha_{r-1} \\ &= \alpha_r + \dots + \alpha_m. \end{aligned}$$

Therefore,

$$\limsup_{n \rightarrow \infty} \frac{\lambda_r^{(m)}(n)}{n} \leq \alpha_r \quad \text{almost sure,}$$

which (together with (7.9)) implies the desired convergence. \square

7.4. Finitely many dual variables, and column lengths. Now let us consider the evolution of an interlacing particle array $\{\tau_j^{(m)}(n)\}$, where $m = 1, \dots, \mathbf{b}$, $j = 1, \dots, m$, which depends on parameters $\{\beta_i\}_{i=1}^{\mathbf{b}}$ satisfying (7.4) (see §7.1.2 and Fig. 13). We can assume that all α_j 's are zero, see §7.1. Under this evolution, only the rightmost particles $\tau_1^{(m)}$, $m = 1, \dots, \mathbf{b}$, can jump independently. Moreover, they form a Markov chain, cf. Remark 7.2. Note that the bottommost particle $\tau_1^{(1)}$ is the slowest one. We will prove

The main goal of this section is to prove Proposition 7.11.

Lemma 7.10. *The following convergence holds for every $i = 1, \dots, \mathbf{b}$:*

$$\frac{\tau_1^{(i)}(n)}{n} \rightarrow m^{\alpha; \beta}(\hat{i}) = \frac{\beta_{\mathbf{b}+1-i}}{1-t}, \quad n \rightarrow \infty, \quad \text{almost surely.}$$

Proof. The key point in the proof is to bound the quantities $\tau_1^{(i)}(n)$ from above, which is achieved by means of their *coupling* with the dynamics $\{Q_i(n)\}_{i=1}^b$ from §7.2. This coupling is more complicated than that of Proposition 7.9 (cf. Remark 7.5). We will put both dynamics $\{\tau_i^{(m)}(n)\}$ (on interlacing arrays) and $\{Q_i(n)\}$ on the same probability space Ω'_n (with n being arbitrary), such that

$$\tau_1^{(i)}(n) \leq Q_i(n) \quad \text{everywhere on } \Omega'_n. \quad (7.10)$$

We are assuming that both dynamics depend on the same parameters $\{\beta_i\}_{i=1}^b$ satisfying (7.4).

We will construct the desired probability spaces Ω'_n by induction on n . Initially, for $n = 0$, all particles $\{\tau_i^{(m)}(0)\}$ and $\{Q_i(0)\}$ are at zero, and Ω'_0 consists of one point. At time n , each point in the space Ω'_n can be realized as a sequence of n triples (ξ_k, η_k, ζ_k) , $k = 1, \dots, n$, plus the whole history of the dynamics of the interlacing array

$$\{\tau_i^{(m)}(k) : m = 1, \dots, b; i = 1, \dots, m; k = 1, \dots, n\}.$$

Here $w = \xi_1 \dots \xi_n$, where $\xi_i \in \{\hat{1}, \hat{2}, \dots, \hat{b}\}$, is the input word of the sampling algorithm (§6.5). If $\xi_n = \hat{i}$ at any time n , then $\eta_n \in \{1, \dots, b - i - 1\}$ encodes the number of particles of the form $\tau_1^{(j)}$, $j > i$, which are pushed (during time step $n - 1 \rightarrow n$) by the jump of $\tau_1^{(i)}$.²⁵ The quantity $\zeta_n \in \{+, -\}$ encodes the event of pushing in the dynamics of $\{Q_j\}$ (during time step $n - 1 \rightarrow n$). Namely, if $\xi_n = \hat{i}$, then $\zeta_n = +$ means that all particles Q_j , $j > i$, were pushed, and $\zeta_n = -$ means that no such particles were pushed. See Fig. 14. We will denote elements of Ω'_n by (ξ, η, ζ, τ) .

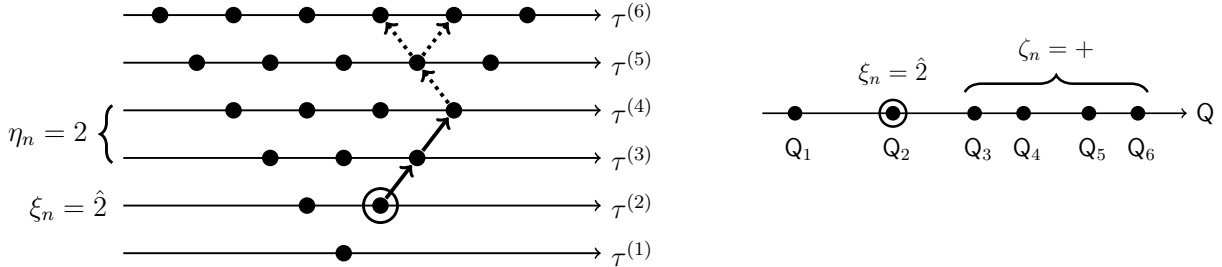


FIGURE 14. Quantities (ξ_n, η_n, ζ_n) . Particles $\tau_1^{(2)}$ and Q_2 jump together; simultaneously particles $\tau_1^{(3)}$ and $\tau_1^{(4)}$ (but not $\tau_1^{(5)}$) are pushed in the dynamics $\{\tau_i^{(m)}\}$, and all particles Q_3, Q_4, \dots are pushed in the dynamics $\{Q_i\}$.

We will let (random) interactions between particles in the bulk of the interlacing array (i.e., all particles except the rightmost ones $\tau_1^{(i)}$) to be independent of the dynamics $\{Q_i(n)\}$, so it suffices to consider only the projection of Ω'_n to (ξ, η, ζ) .

Employing our induction on n , let us assign a *joint* probability distribution to the triple (ξ_n, η_n, ζ_n) which is added during time $n - 1 \rightarrow n$. First, we want both dynamics to receive the

²⁵Note that once a particle $\tau_1^{(m)}$, for some $m > i$ was not pushed, all upper particles $\tau_1^{(j)}$ ($j > m$) also cannot be pushed, see Remark 6.20.

same “jumping signal” ξ_n with probability $P(\xi_n = \hat{i}) = m^{\alpha;\beta}(\hat{i}) = \beta_{b+1-i}/(1-t)$ (this part is similar to the proof of Lemma 7.8).

Now, conditioning on $\xi_n = \hat{i}$ for some fixed \hat{i} , let us denote by r_j , $j = 0, 1, 2, \dots$ the *marginal* probability that $\eta_n = j$, and by r_{\pm} the *marginal* probability that $\zeta_n = \pm$. (It is worth noting that these probabilities depend on the state of the corresponding dynamics at time $n-1$, which is in turn determined by the history $(\eta_1, \dots, \eta_{n-1})$ or $(\zeta_1, \dots, \zeta_{n-1})$, respectively.) Let us choose $u = 0, 1, 2, \dots$ such that

$$r_0 + r_1 + \dots + r_{u-1} \leq r_-, \quad r_0 + r_1 + \dots + r_u > r_-, \quad (7.11)$$

and assign the following joint probability weights (recall that they are conditional on $\xi_n = \hat{i}$):

$$\begin{aligned} P(\eta_n = j, \zeta_n = -) &= r_j, & j = 0, 1, \dots, u-1; \\ P(\eta_n = u, \zeta_n = -) &= r_-; \\ P(\eta_n = u, \zeta_n = +) &= r_0 + r_1 + \dots + r_u - r_-; \\ P(\eta_n = j, \zeta_n = +) &= r_j, & j = u+1, u+2, \dots \end{aligned} \quad (7.12)$$

All other joint probabilities are set to zero. Clearly, thus defined probability distribution on (ξ_n, η_n, ζ_n) projects in a desired way to (ξ_n, η_n) (which corresponds to the dynamics on interlacing arrays) and to (ξ_n, ζ_n) (which corresponds to the auxiliary dynamics $\{Q_i\}$).

Formulas (7.12) can be interpreted as follows: if $\geq u$ of the rightmost particles is pushed in the dynamics on interlacing arrays, then the pushing event happens in the auxiliary dynamics $\{Q_j\}$ as well.

Now let us prove the inequalities (7.10). By induction, assume that they hold at time n . Let $\xi_{n+1} = \hat{i}$ for some \hat{i} . Consider the following cases:

- $Q_{i+\ell}(n) > \tau_1^{(i+\ell)}(n)$ for each $\ell = 0, 1, 2, \dots$. Then, since during the time $n \rightarrow n+1$ all coordinates can increase at most by one, the inequalities continue to hold at time $n+1$.
- $Q_i(n) > \tau_1^{(i)}(n)$, and for some $\ell > 0$ we have $Q_{i+\ell}(n) = \tau_1^{(i+\ell)}(n)$ (assume that this ℓ is the smallest among all indices with this second property). Then, in the notation before (7.11), we clearly have

$$r_+ \geq t^{Q_{i+\ell}-Q_i}, \quad r_{\ell} + r_{\ell+1} + r_{\ell+2} + \dots = t^{\tau_1^{(i+\ell)} - \tau_1^{(i)}}.$$

Therefore,

$$r_+ \geq r_{\ell} + r_{\ell+1} + r_{\ell+2} + \dots,$$

so $u \leq \ell$, where u is defined in (7.11). This implies that if the particles $\tau_1^{(i+1)}, \dots, \tau_1^{(i+\ell)}$ (and maybe some of the particles $\tau_1^{(j)}$, $j > i+\ell$, as well) were pushed, then the pushing event also happened in the auxiliary dynamics, so all the particles Q_j , $j > i$, were pushed. This readily implies that the desired inequalities continue to hold at time $n+1$.

- $Q_i(n) = \tau_1^{(i)}(n)$. Then also $Q_{i+1}(n) \geq \tau_1^{(i+1)}(n)$ by the induction hypothesis, and thus $r_0 \geq r_-$ in the notation before (7.11). By (7.12), this means that if the particle $\tau_1^{(i+1)}$ (and maybe some of the particles $\tau_1^{(j)}$, $j > i+1$, as well) was pushed, then the pushing event also happened in the auxiliary dynamics $\{Q_j\}$. Thus, inequalities (7.10) continue to hold at time $n+1$ as well.

The above cases show that (7.10) holds.

The desired limiting bound from below is achieved with the help of (7.10) and Proposition 7.6 in exactly the same way as in the end of the proof of Lemma 7.8. This completes the proof of the claim. \square

Proposition 7.11. *The following Law of Large Numbers holds for every $m = 1, \dots, \mathbf{b}$ and every $i = 1, \dots, m$:*

$$\frac{\tau_i^{(m)}(n)}{n} \rightarrow \frac{\beta_i}{1-t}, \quad n \rightarrow \infty, \quad \text{almost surely.} \quad (7.13)$$

Proof. Since $\tau_i^{(m)} \leq \tau_1^{(m+1-i)}$ for $i = 1, \dots, m$ due to interlacing, this proposition follows from Lemma 7.10 by induction in exactly the same way as Proposition 7.9 follows from Lemma 7.8. \square

7.5. Completing the proof. With the results of §7.3 and §7.4, we have now proved Theorem 7.1 in the case when both sequences α and β are finite. Let us now extend this statement to the general case when these sequences are allowed to be infinite, so the specialization depends on

$$\alpha_1 \geq \alpha_2 \geq \dots \geq 0, \quad \beta_1 \geq \beta_2 \geq \dots \geq 0, \quad \sum_{i=1}^{\infty} \alpha_i + \frac{1}{1-t} \sum_{i=1}^{\infty} \beta_i = 1. \quad (7.14)$$

Assume that $\lambda(n)$ is the Young diagram distributed according to the measure $\mathcal{HL}_n^{\alpha; \beta; \mathbf{Pl}_0}$. Recall that by $\lambda_j(n)$ and $\lambda'_j(n)$ we denote row and column lengths of this diagram.

Lemma 7.12. *For any $k = 1, 2, \dots$, we have*

$$\liminf_{n \rightarrow \infty} \frac{\lambda_k(n)}{n} \geq \alpha_k, \quad \liminf_{n \rightarrow \infty} \frac{\lambda'_k(n)}{n} \geq \frac{\beta_k}{1-t}.$$

Proof. Let us fix k and prove that $\liminf_{n \rightarrow \infty} \frac{\lambda_k(n)}{n} \geq \alpha_k$. Consider the following ordering of the alphabet \mathcal{A} :

$$1 < 2 < \dots < k < \text{rest of the letters (ordered arbitrarily)}.$$

The sampling algorithm for the measure $\mathcal{HL}_n^{\alpha; \beta; \mathbf{Pl}_0}$ under this ordering (§6.4–§6.5) is a Markov dynamics with state space consisting of towers of Young diagrams. First k floors in such a tower constitute an interlacing particle configuration $\{\lambda_i^{(m)}(n)\}_{1 \leq i \leq m \leq k}$. Moreover, employing the bijection with \mathcal{A} -tableaux, we see that $\lambda_k^{(k)}(n)$ is the number of letters k in the k -th row of the Young diagram $\lambda(n)$ (this diagram is the shape of the \mathcal{A} -tableau). Therefore, $\lambda_k(n) \geq \lambda_k^{(k)}(n)$.

On the other hand, note that the behavior of the first k floors does not depend on what is happening above them. Therefore, we are in a position to apply Proposition 7.9 to the first k floors (formally, we can consider the dynamics with finitely many α -parameters $\alpha_1, \alpha_2, \dots, \alpha_k$, and $1 - \sum_{i=1}^k \alpha_i$, and apply Proposition 7.9 to this dynamics; the distribution of the first k floors will be the same). We obtain that $\lambda_k^{(k)}(n)/n \rightarrow \alpha_k$.

The corresponding statement about column lengths follows by considering the ordering

$$\hat{1} < \hat{2} < \dots < \hat{k} < \text{rest of the letters (ordered arbitrarily)},$$

and referring to Proposition 7.11; again, we are able to apply this proposition due to the fact that the behavior of the first k floors does not depend on what is happening above them. \square

To finish the proof of Theorem 7.1, it now remains to establish upper bounds corresponding to the lower bounds of Lemma 7.12:

Lemma 7.13. *For any $k = 1, 2, \dots$ and any $\varepsilon > 0$, we have*

$$\limsup_{n \rightarrow \infty} \frac{\lambda_k(n)}{n} < \alpha_k + \varepsilon, \quad \limsup_{n \rightarrow \infty} \frac{\lambda'_k(n)}{n} < \frac{\beta_k}{1-t} + \varepsilon.$$

Proof. Let us prove the bound for the row lengths (the case of the column lengths is analogous). We argue similarly to the proof of Proposition 7.9.

Fix k and ε . Using (7.14), choose \mathbf{a} and \mathbf{b} so large that $\mathbf{a} > k$ and that

$$\sum_{i=1}^{\mathbf{a}} \alpha_i + \sum_{j=1}^{\mathbf{b}} \frac{\beta_j}{1-t} > 1 - \varepsilon.$$

Moreover, we know that

$$\sum_{i=1}^{\mathbf{a}} \lambda_i(n) + \sum_{j=1}^{\mathbf{b}} \lambda'_j(n) \leq n + \mathbf{a}\mathbf{b},$$

because the Young diagram $\lambda(n)$ has n boxes, and in the summation over i and j above we can count twice only the boxes from first \mathbf{a} rows and \mathbf{b} columns.

Therefore, we can write (omitting dependence on n in λ_j and λ'_j)

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{\lambda_k(n)}{n} &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \left(n - \mathbf{a}\mathbf{b} - \lambda_1 - \lambda_2 - \dots - \lambda_{k-1} - \lambda_{k+1} - \dots - \lambda_{\mathbf{a}} - \lambda'_1 - \dots - \lambda'_{\mathbf{b}} \right) \\ &\leq 1 - \liminf_{n \rightarrow \infty} \frac{1}{n} \left(\lambda_1 + \dots + \lambda_{k-1} + \lambda_{k+1} + \dots + \lambda_{\mathbf{a}} + \lambda'_1 + \dots + \lambda'_{\mathbf{b}} \right) \\ &\leq \alpha_k + 1 - \sum_{i=1}^{\mathbf{a}} \alpha_i - \sum_{j=1}^{\mathbf{b}} \frac{\beta_j}{1-t} < \alpha_k + \varepsilon. \end{aligned}$$

Here in the last estimate for \liminf we have used Lemma 7.12. □

Lemmas 7.12 and 7.13 readily imply Theorem 7.1.

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